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THE

ELEMENTS OF SOLID GEOMETRY.

THE

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OF

SOLID GEOMETRY

BY

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PREFACE.

IN composing the treatise here offered to the student of Elementary Geometry, it has been my principal object to present the subject as a logically coherent body of propositions obtained by the simplest lines of deduction from postulates explicitly stated, and to extend its scope beyond the rather meagre range of the first twenty-one propositions of Euclid's Eleventh Book, which has long been the customary limit of the young student's formal study of Solid Geometry. With this object I have found it necessary to deviate considerably from Euclid's sequence of propositions and largely to modify his demonstrations. At the same time, as the deviations will, I believe, be found to be in the direction of simplification by bringing the propositions nearer, along the line of deduction, to the fundamental postulates, I think that the student who gets up the subject from this book will, even as regards mere examination purposes, be at no disadvantage from not having studied the text of Euclid, as it appears in the usual editions.

Many of the demonstrations in Euclid's Eleventh Book, notably those of Props. 4, 6, and 8, are highly artificial; and I believe this to be mainly due to his dealing with perpendiculars before parallels and intersections, and treating the latter as dependent on the former. In Section I. of this work I have treated of Intersections and Parallels, and then in Section II. proceeded to deal with Normals and Obliques. In the latter I have assumed an axiom, which I have not thought it necessary to state, because it is a *general* axiom, applicable to magnitudes of all kinds, and not specially a postulate of Geometry, and because I hold that it is not logically necessary, but only a question of convenience, whether such general axioms or common notions (the κοινὰι ἔννοιαι of Euclid) should be explicitly stated, while logical coherence requires that every special postulate should be distinctly enunciated. The axiom may be thus stated: "If a magnitude be such that it cannot be made to vanish, there is a finite value, or a set of equal finite values, of that magnitude less than any other values." From this I establish the existence of a normal at every point of a plane, and the properties of normals then directl follow. I would invite attention to the simplicity attaine by this mode of treatment, and to facilitate comparison have added to each proposition a reference to the cor sponding proposition in Euclid.

In Section III. I have treated of Dihedral and Solid Angles, and in Section IV. of Polyhedra, and in particular of the five Regular Solids, with the definitions and a summary of the leading characteristics of the Prism, Cylinder, Pyramid, Cone, and Sphere.

In Section V. the Volumes of Solids are considered, and the usual rules or formulæ for the numerical measures of volumes in cubic units deduced. To this Section I have added an Appendix on Symmetry, which I hope will be useful in assisting the student to clearer conceptions of figures and their relations in solid space. A short word has long been wanted for the decasyllabic "rectangular parallelepiped": I am sanguine that the word "cuboid," which I have here introduced, will meet with acceptance from mathematicians.

In the concluding Section (VI.) I have given a sketch of the leading elementary propositions of Spherical Surface Geometry, partly as throwing light on, and suggesting convenient modes of treatment of problems in Solid Geometry, and as leading up to Spherical Trigonometry, but mainly with a view to a comparison with corresponding propositions in Plane Geometry.

I have prefixed to the whole a "Preliminary Discussion on the Postulates of Geometry." It is in my opinion highly desirable that the student should at some point in his course examine carefully into the fundamental assumptions of Geometry, and the logical affiliation of its propositions. This is clearly impossible till he has attained to some familiarity with the subject, and it can hardly be done satisfactorily till he is prepared to enter on the study of Solid Geometry; but he may, and probably in most cases will, postpone it till a later period: accordingly I have made the rest of the book quite independent of this discussion, which may be read or omitted by the student without prejudice to his clear understanding of the propositions in the Sections which follow.

It will be seen that I have, in accordance with my title, strictly confined myself to the *Elements* of Solid Geometry, to the exclusion of the higher methods and results of Modern Geometry. These may best be studied in separate treatises, such as the "Projective Geometry" of Cremona, rather than in detached fragments introduced into a book whose main method is Euclidean, but to such treatises I have hopes that the present may form a useful introduction.

I may add, in conclusion, that this treatise has been developed out of a "Syllabus of Solid Geometry," which I submitted to a Committee of the Association for the Improvement of Geometrical Teaching, and which that Committe received with a considerable amount of favour. Findin that it was difficult to make clear the intended thread connection in the Syllabus by the bare enunciation of

Theorems in a certain sequence, I was led to write out the demonstrations, and then it seemed to me desirable to go further than I had originally proposed, and make a treatise, which might be considered as fairly complete, within the limits of Elementary Geometry, for the young student. In making this statement I must not be understood to claim for my work that it is issued with the sanction or authority of the Association or of the Committee referred to above. I am alone responsible for the entire contents, though I gratefully acknowledge that I have profited by the criticisms on my Syllabus of certain members of the Committee.

R. B. H.

HARROW, *April*, 1890.

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ELEMENTS OF SOLID GEOMETRY.

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THE
ELEMENTS OF SOLID GEOMETRY.

PRELIMINARY DISCUSSION OF THE POSTULATES OF GEOMETRY.

THE student of Geometry, who has acquired some familiarity with the subject, will find it profitable, in order to attain a view of the science as an organic whole logically deduced from a few simple conceptions, to examine and arrange in logical order those fundamental conceptions of space (however derived), which lead to the fundamental postulates[1] (or so-called axioms) of geometry. A suitable occasion for doing this arises, when, passing from plane geometry, he commences the study of geometry in three-dimensional space. It is with the object of facilitating such an examination, that the following short discussion is introduced as preliminary to the sections of this treatise in which the elementary propositions of solid geometry are established.

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[1] I use the word as the equivalent of Euclid's *αἰτήματα*, under which he rightly includes what modern editors, following some ancient commentators, reckon as the 10th, 11th, and 12th Axioms, and under which he might well have included the 8th also. The remainder of the so-called axioms are fitly termed by Euclid, *κοιναὶ ἔννοιαι*,—common notions. The distinction of postulate and axiom as corresponding to that of problem and theorem is not fundamental, and is not Euclid's. His idea seems to have been the more logical one of distinguishing the special assumptions of geometry, every one of which ought to be explicitly stated, from those more general "common notions" (applicable to all kinds of magnitudes) of which he states only those which he most *frequently requires*, without attempting to make the list exhaustive.

It should, however, be understood that this discussion is in no way a *necessary* preliminary to the study of those sections, so that it may be read independently of the rest of the book, or omitted altogether, at the discretion of the student or his teacher.

1. Our conception of space is that of an unlimited extension on all sides.

A finite portion of space regarded as separated from the rest, is termed a *solid.*

That which separates one portion of space, whether finite or infinite, from the rest and which is no part of either, is termed a *surface.* If the portion separated is finite, the surface is a *closed* surface.

That which separates one portion of a surface, whether finite or infinite, from the rest of the surface, and which is not part of either, is termed a *line.* If the portion separated is finite, the line is a *closed* line or curve.

That which separates one portion of a line from the rest, and which is not a part of either, is a *point.*

Of a point we can predicate *position* in space and nothing else. In other words, two points can only differ in, or be compared as to, position. Such terms as *size*, *shape*, &c., are inapplicable to points. A point may be determined as the *end of a terminated* line, or as the intersection of two lines.

Points, *lines*, *surfaces*, and *solids* form the subject-matter of Geometry.

2. The first postulate of Geometry is, that any point, line, surface, or solid, or any combination of these, may be conceived to have its position in space changed without change in any other respect. Or, using the term *figure* in the general sense of any combination of points, lines, surfaces, or solids,

POST. 1. *Any figure may be transferred from one position in space to another without change of shape or size.*

3. If one point of a figure is given in position or fixed, the possible changes of its position are limited, but it is still free to change its position subject to this *one* condition.

If a second point in the figure is also given in position or fixed, the figure is still free to change its position, but in a manner still further limited. The motion, by which the change from one position to another is effected in this case, is called *rotation*. In rotation there is a series of points forming a line passing through the two fixed points, whose position remains unchanged by the motion. This line is termed the *axis* of rotation.

If, farther, a third point in the figure, not in the axis of rotation, is given in position or fixed, the freedom to change its position ceases, and the figure is determined in position or fixed.

In general, then, a figure of given form and size, that is, whose configuration, or the relative positions of whose parts are all given, has its position determined by the positions of *three* of its points.

4. LINES. A point in moving from one position to another passes along, or is said to describe or generate, a line. This

line is said to be the *locus* of the moving point. The line, though generated by the point, is not made up of points, for a point is defined as being of no size and so no part of a line.

The position of one point on a line relatively to another on the same is defined by a *single* element, termed the distance between the points along the line or the length of this part of the line, with the addition of the statement whether this length is measured in one of the two *senses*, forwards or backwards, along the line. For most purposes in elementary geometry, it is not necessary to regard this distinction of *sense* along a line, but in the higher geometry it is of essential importance, and the distinction is denoted by the signs + and −.

Since *one* element (a length or distance) is sufficient to define the position of one point on a line relatively to another, a line is said to be a magnitude of *one dimension*: or, Linear Geometry, which is concerned with points on a given line or lengths measured along the same, is *one-dimensional.*

5. *The Straight Line.* In general the position of a line of given form, as of any figure, is determined, when the positions of *three* points upon it are given, but this does not exclude the possibility of some special line being determined completely by *two* points. Accordingly we may state as the second postulate of Geometry:

POST. 2. *For any two given points there is a line passing through both which is completely determined, both as to form and position, by the two points.*

Such a line is said to be *straight.*

There is *only one* straight line determined by two points: for if there were more than one, since by the postulate each is *completely* determined by the two points and by nothing else, there would be nothing to distinguish between such lines, or they would be identically the same.

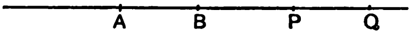
A straight line must be indefinitely extended in both senses, for to limit it would be to introduce another condition for its determination.

Any portion of a straight line between two points on it is usually termed a *finite straight line*,[1] or, simply, a straight line.

The straight line through two points in a figure, being completely determined by the two points, remains unchanged in position, while the figure rotates about the two points fixed. Hence an axis of rotation is a straight line.

All the parts of a straight line are identically alike, or any one part is superposable on any other part so as to coincide with it.

For take the straight line passing through the points A, B, and let the line be moved and replaced on A, B, in a new position so that the point P in it falls on A, and Q on B, then the line must by the postulate occupy the same position in space as before, since it is completely determined



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[1] On account of the importance of distinguishing in words between the infinite whole of a straight line and a finite length on a straight line, some authors call the part of a straight line between two points the *join* of those points. The term has hardly been generally accepted, and the author would suggest the word *link* as preferable both in sound and sense.

by A and B alone; therefore the part between P and Q must be exactly like the part at first between A and B, so that either of these parts if placed on the other would coincide with it.

The shortest distance between two points is that along the straight line passing through them.

Assuming[1] that only one distance can be measured between two points less than any other distance between them, that distance must be that measured along the straight line joining the points; for, if measured on any other line, the position of that line could be altered by rotation about the two points, so that the same distance could be measured on more than one line between the points.

From the foregoing discussion, the usual assumptions as to the straight line obviously follow, namely: —

That one straight line and only one can be drawn between two points and produced indefinitely beyond those points. (*Euc.* Post. 1, 2; *Pl. G.*[2] Ax. 2.)

That two straight lines cannot enclose a space. (*Euc.* Post. 5 in some MSS., Ax. 10 of modern editors.)

That two straight lines which meet in one point cannot meet again unless they coincide. (*Pl. G.* Ax. 2, β.)

6. *Curves.* Lines not straight are termed *curved lines* or *curves*, and are determined as to form and position by not

---

[1] This assumption ought perhaps to be stated as a distinct postulate, though it would seem to be involved in the notion of the least distance between two points being dependent on the positions of the two points in space and on *nothing else.*

[2] *The Elements of Plane Geometry*, or the *Syllabus* of the same, published by the Association for the Improvement of Geometrical Teaching, will be in this book referred to as Pl. G.

less than three of the points through which they pass. The only curve determined completely by three points (not in the same straight line) is the circle.

7. SURFACES. If two points are taken on a surface, of all the lines that can be drawn on the surface through the two points there must always be one, and in general only one, whose length is less than that of any other. (Thus a fine thread drawn tight between points on a surface will coincide with a particular line on the surface, and in general with no other. In the case of a sphere this line is a great circle of the sphere.) Such a line may be termed the *direct*[1] line on the surface through the two points. In certain cases there may be points so related that any number of direct lines can be drawn between the two points, as in the case of the direct lines (or great circles) joining the opposite extremities of a diameter of a sphere, but these cases are exceptional and confined to particular surfaces and to points on such surfaces specially related to one another, while for all other pairs of points there is only one direct line passing through them.

The position of one point on a given surface relatively to another point on the same requires *two* independent elements to define it, which may be taken as the length of the direct line between the points and the direction of that line, or the angle which it makes with some definite line through the second point. Since these two elements (or two others equivalent to them) are necessary and sufficient to determine the relation between two points on a given surface, a surface is said to be a magnitude of *two dimensions*, and the geometry of points, lines, and figures on a given surface is *two-dimensional.*

---

[1] In the higher geometry such a line is termed a *geodesic* line. A simpler term seems desirable, at any rate in elementary geometry.

8. *The Plane.* If between two points on a surface, a straight line can be drawn without leaving the surface, this, being the shortest possible distance between the two points, must be the *direct* line on the surface through those points. This leads to the following postulate:

POST. 3. *There is a surface such that every direct line between two points upon it is straight.*

Such a surface is termed a *plane surface,* or, simply, a *plane.* This postulate is usually assumed in the definition of a plane which we have adopted in Section I., namely:

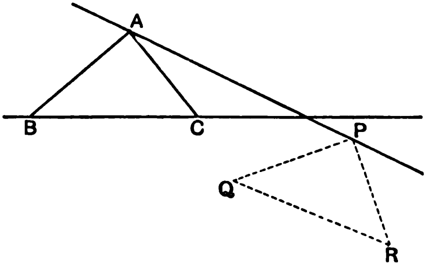
A *plane*[1] is a surface such that, if any two points are taken in it, the straight line passing through those points lies wholly in the surface.

A plane, like the straight lines in it, extends indefinitely in all directions, separating the whole of space into two infinite parts.

A plane is *completely* determined by three points in it, provided they are not in the same straight line; or in other words, there is one, and only one, plane passing through three given points, not in the same straight line. This property follows from the definition above given. For if A, B, C be the three points, the straight line through B, C lies in the plane, and divides the infinite extent of the plane into two infinitely extended parts, and if *P* be any point in the plane, the straight line through *A* and *P*, being in the plane by definition, must meet that through *B* and *C* (except only in the case of parallelism); hence the whole plane would be generated by a straight line through *A* moving so as to pass through every point in that through *B* and *C*. The excepted case of the points on the parallel to *BC* is met by supposing the line to pass through *B*, and move along that through *A* and *C*. This property, which is true of no other surface, might have been taken as the definition of a plane, analogous to that given above for the straight line.

---

[1] This is not Euclid's definition, though it is the one adopted in modern editions of Euclid. As he defines a straight line as that which *ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται,* so he defines a plane as the surface which *ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς ἐυθείαις κεῖται.* Whatever the exact force of these words may be, they are useless as a definition, since they are merely verbal, defining one term by another not more intelligible than the first. The latter definition tacitly assumes that there are *straight lines on* the plane.



All the parts of a plane are identically like, or any one part is superposable on any other part so as to concide with it. For if the plane through the points *A*, *B*, *C* be moved and then replaced on *A*, *B*, *C* so that another point *P* in it falls on *A*, *Q* on *B*, and *R* on *C*, the plane being defined completely by *A*, *B*, *C* must occupy the same position as before, and then the part of it within the triangle *PQR* exactly concides with the part within *ABC*.

9. The two-dimensional geometry of lines and points in a plane is known as Plane Geometry.

The postulates of plane geometry are those already stated, together with that relating to parallels, which may be stated in the following form:

POST. 4.[1] *Through the same point not more than one line can be drawn in the same plane with a given line which does not meet that line.*

To this must be added, as the general postulate for the metrical comparison of all geometrical magnitudes (plane or solid), or the fundamental test of equality in geometry:

POST. 5.[2] *Figures, which can be brought into coincidence by superposition, are equal to one another.*

10. In pure plane (or surface) geometry, in which figures are limited to remain always in the plane (or surface), a further postulate becomes necessary, in order to assert that figures, which are equal line for line and angle for angle, are equal as wholes, though they may not be superposable as wholes. Such a postulate would be this:

Figures on a plane (or other surface) which are symmetrical with respect to a straight (or direct) line in it are equal to one another. (See Discussion on Symmetry, p. 98.)

The slight sketch of the surface geometry of the sphere, in Section VI., will illustrate this.

11. SOLID SPACE. In solid space the position of one point relatively to another requires for its determination *three*

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[1] If this postulate is not admitted and it is considered possible to have more than one line through a point in a plane not meeting a given line in the same, a system of non-Euclidean geometry results, definite and logically coherent, as was first shown by Lobatschewsky, Professor of Mathematics in the University of Kasan, about the year 1840.

[2] For a full discussion of this postulate, see Clifford's lecture on the "Philosophy of the Pure Sciences," *Lectures and Essays*, p. 222.

independent elements, and hence the geometry of solid space is said to be *three-dimensional.*[1]

The particular elements (or *coordinates*), by which the relative position of two points is defined, may be taken in various ways. Thus if through one of the points three fixed straight lines are taken not in the same plane, the position of the other may be defined by its distances in the directions of each of the three lines from the plane passing through the other two. If one only of these distances is given, the point is limited to lie in a plane parallel to the other two: if two of them are given, it is limited to lie in a straight line parallel to the third: if all three are given, the particular point in this line is determined. These, and other methods of defining the positions of points in space are fully developed in treatises on coordinate geometry of three dimensions.

12. The foregoing discussion may be summed up by repeating the postulates, to which it has led:

POST. 1. *Any figure may be transferred from one position in space to another without change of shape or size.*

POST. 2. (THE STRAIGHT LINE.) *For any two given points there is a line passing through both, which is completely determined, both as to form and position, by the two points.*

POST. 3. (THE PLANE.) *There is a surface such that every direct line between two points upon it is a straight line.*

POST. 4. (PARALLEL LINES.) *Through the same point not more than one line can be drawn in the same plane with a given line which does not meet that line.*

---

[1] It would be out of place in an elementary treatise to do more than allude to the speculations of mathematicians and others as to space of more than three-dimensions, since our experience and the resulting conceptions are limited to three-dimensional or solid space.

POST. 5. (TEST OF EQUALITY.) *Figures, which can be brought into coincidence by superposition, are equal to one another.*

To which should be added, as the test of equality for non-superposable figures (see Discussion on Symmetry, pp. 98 to 108):

POST. 6. *Figures which are completely symmetrical with respect either to a point, or to a straight line, or to a plane, are equal to one another.*

SECTION I.

INTERSECTIONS AND PARALLELS.

The following postulate is usually tacitly assumed in elementary geometry, but may with advantage be explicitly stated.

POST. 1. *Any assemblage of points, lines, surfaces, or solids may have its position in space changed without any change in the relative positions of its parts.*

Or, (using the term *figure* for such assemblage)

Any figure may be transferred from one position in space to another without change in shape or size.

Def. 1. A *plane* is a surface such that, if any two points are taken in it, the straight line passing through these points lies wholly in the surface.

A plane must be understood as indefinitely extended in all directions, as are the straight lines which lie in it.

A limited portion of a plane forms a *plane figure.*

Def. 2. Straight lines which are in the same plane and do not meet (however far extended) are said to be *parallel.*

Def. 3. A straight line, which does not meet a plane (however far extended), is said to be *parallel* to the plane.

Def. 4. Planes, which do not meet (however far extended), are said to be *parallel.*

THEOREM I. (including Euc. XI. 2).

A plane is determined by

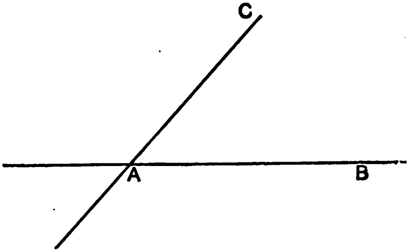
either (1) *three points, not in the same straight line ;*

or (2) *a straight line and one point not in that line ;*

or (3) *two straight lines which either meet or are parallel.*

Let *A*, *B*, *C* be three points not in the same straight line. Then any plane may be so changed in position as to pass through *A*, next moved about *A* fixed until it passes through *B*, and finally rotated about *A* and *B* fixed until it passes through *C*; it is then fixed in position, and, being completely determined by the points *A*, *B*, *C*, may be denoted as the plane *ABC*.

Also, since the straight line through *A* and *B* lies in the plane, (Def. 1.)
the plane is completely determined by the line through *AB* and the point *C*, and may be described as the plane through the line *AB* and the point *C*.



Further, since the straight line through *A* and *C* lies in the plane, the plane is completely determined by the two straight lines AB, AC intersecting at A, and may be described as the plane through the lines AB, AC.

Lastly, if two lines are parallel, they are (by Def.) in the same plane, and that plane is determined when the lines are given, so that it may be described as the plane through the two parallel lines. Q. E. D.

COR. 1. A triangle is the only rectilineal figure that is necessarily plane.

COR. 2. A plane may be generated by a straight line, which always passes through a fixed point and intersects a fixed straight line which does not pass through that point. Also by a straight line which always passes through two fixed straight lines, which intersect or which are parallel.

THEOREM II. (including Euc. XI. 3).

If a plane and a straight line meet, they intersect in one point, and if two planes meet, they intersect in one straight line.

For, if a straight line through a point A without a plane meet that plane in the point B, it cannot meet it again, since, if it did, it would lie wholly in the plane. (Def. 1.)

Also, if A, B are two points in the line of intersection of two planes, the straight line through A and B lies in both planes, (Def. 1.)

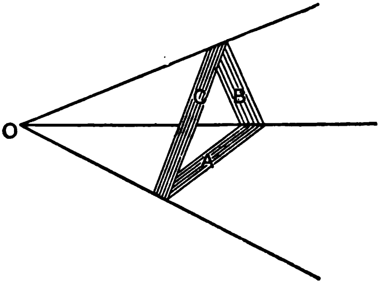
and any other point C in their line of intersection must be in this line, for otherwise the two planes would coincide: (Th. I.)

hence the planes intersect in one straight line. Q. E. D.

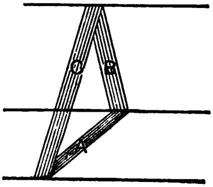
THEOREM III.

If three planes intersect each with the other two, the three lines of intersection either meet in one point or are parallel.

Let A, B, C be three planes; then the lines of intersection of B and C, of C and A, and of A and B either meet in a point or are parallel.



If the intersection of B and C meet the plane A in O:
then, since O is in both the planes A and B,
the intersection of A and B passes through O;
and since O is in both the planes A and C,
the intersection of A and C passes through O:
therefore the three lines of intersection meet in O.



If the intersection of B and C is parallel to the plane A: then, supposing that the intersections of A and B and of A and C met in the point O,

O would be a point in the intersection of B and C,

so that the intersection of B and C would meet the plane A:

hence the intersections of A and B and of A and C do not meet,

and therefore, being in the plane A, they are parallel:

(Def. 2.)

also, being in the plane A, they cannot meet the intersection of B and C, which is parallel to that plane:

therefore the three intersections in this case are parallel.

Q. E. D.

COR. If two intersecting planes pass through two parallel lines, their intersection is parallel to these lines.

THEOREM IV. (Euc. XI. 16).

The intersections of a plane with two parallel planes are parallel.

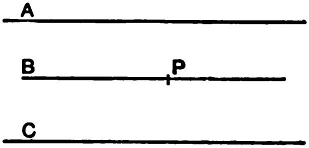
For these intersections do not meet, since, if they did, the two planes would meet; whereas, being parallel, they do not meet.

Further, they are in the same plane, therefore they are parallel. (Def. 2.)

Q. E. D.

Theorem V. (Euc. XI. 9).

Lines parallel to the same straight line are parallel to one another.



Let the lines A and B be both parallel to C; then A and B are parallel.

Take a plane through A and any point P in B.

Then the planes of the parallels A, C and B, C and the plane through A and the point P are three planes, each intersecting the other two,

the lines of intersection being A, C, and a line through P,

and since A, C are parallel, the line through P is parallel to both; (Th. III.)

but there cannot be more than one line through P parallel to C, and B is parallel to C, (Pl. G.[1] Ax. 3.)

therefore the line through P must coincide with B:

therefore A and B are parallel.

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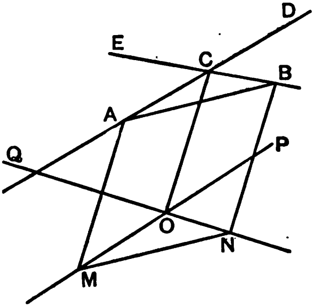
[1] The reference is to *The Elements of Plane Geometry*, or the *Syllabus* of the same, published by the Association for the Improvement of Geometrical Teaching, which will be always referred to as Pl. G.

THEOREM VI. (Euc. XI. 15 and 10).

If two intersecting lines are parallel respectively to two other intersecting lines, the plane of the first pair is parallel to the plane of the other pair, and the angles between the first pair equal respectively to the angles between the other pair.



Let AD, BE, which intersect at C, be respectively parallel to MP, NQ, which intersect at O;
then the plane ACB is parallel to the plane MON,
and the angle ACB equal to the angle MON,

A, M being on the same side of CO in the plane ACO, and B, N being on the same side of CO in the plane BCO.

First. The planes ACB, MON are parallel.

For, if the planes ACB, MON meet,
their intersection is parallel to AD and MP,
since these lines, through which the planes pass, are parallel

(Th. III., Cor.)

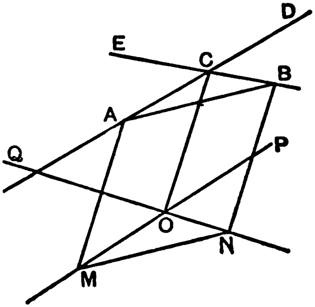
C 2

and their intersection is parallel to *BE* and *NQ*, sinc these lines also, through which the planes pass, are parallel; therefore the same line is parallel to both *AD* and *BE*, which is impossible, since *AD* and *BE* intersect.

Therefore the planes *ACB*, *MON* do not meet; that is they are parallel. (Def. 4)

Further, the angles *ACB*, *MON* are equal.

Join *CO*, and let a plane parallel to *CO* cut the plane o the parallels *AD*, *MP* in *AM*, and that of the parallels *BE*, *NQ* in *BN*: join *AB*, *MN*.



Then, because the plane *AMNB* is parallel to *CO*,

AM, *BN*, *CO* are parallel: (Th. III.)

therefore, *AC*, *MO* being parallel, and also *BC*, *NO*,

CM and *CN* are parallelograms:

therefore *AM*, *BN* are each equal and parallel to *CO*,

and *AM* is equal and parallel to *BN*.

Hence *AMNB* is a parallelogram. (Pl. G. I. 31.)

Then in the triangles ABC, MNO,

AB, BC, CA, are respectively equal to MN, NO, OM,
(Pl. G. I. 29.)

therefore the angle ACB is equal to the angle MON.
(Pl. G. I. 18.)

Also since ACB is equal to DCE and MON to POQ, either of the angles ACB or DCE is equal to either of the angles MON or POQ.

In like manner either of the angles ACE or BCD is equal to either of the angles MOQ or NOP. Q. E. D.

NOTE.—Two parallel lines are said to have the same *direction*, but if they are drawn *from* (not *through*) two points, they may be either both on the same side of the line through the two points or one on one side and the other on the other. It is convenient to use the word "*sense*" (as French mathematicians use the word "*sens*") to denote[1] likeness or difference in this respect, and to say, that parallel lines drawn from two points both on the same side of the line joining those points have the *same* or *like senses*, while those drawn the one on one side and the other on the other have *opposite senses*. Then the preceding proposition may be more completely and definitely enunciated thus:

If two lines which meet are parallel (or have the same directions) respectively with two other lines which meet, the angle between the first pair will be equal to the angle between the other pair, when (Fig. 1) *both pairs of parallels have like,*

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[1] A complete discussion of "sense," with the use of the signs + and − to denote opposite senses, would be out of place here. The above is sufficient for the purposes of Elementary Geometry.

or both opposite, senses; but supplementary, when (Fig. 2) the parallels of one pair have like, and those of the other pair opposite, senses.

Fig. 1.

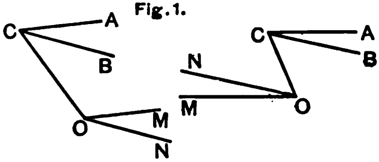
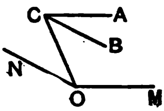
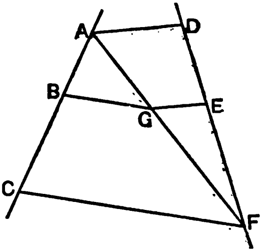


Fig. 2.



THEOREM VII. (Euc. XI. 17.)

If two straight lines are cut by parallel planes, the segments of the one have to one another the same ratio as the corresponding segments of the other.



Let the straight lines *ABC*, *DEF* meet three parallel planes in *A*, *B*, *C* and *D*, *E*, *F* respectively:

then $AB : BC :: DE : EF$.

Join *AF*, and let *G* be the point where *AF* meets the plane though *D*, *E*; join *BG*, *GE*, *AD*, *CF*.

Then because two parallel planes are cut by the plane ACF in the lines BG, CF,

BG is parallel to CF, (Th. IV.)

and therefore $AB : BC :: AG : GF$. (Pl. G. IV. 15.)

In like manner because two parallel planes are cut by the plane AFD in the lines GE, AD,

GE is parallel to AD,

and therefore $AG : GF :: DE : EF$.

Hence, the ratios $AB : BC$ and $DE : EF$ being each equal to the ratio $AG : GF$,

$AB : BC :: DE : EF$. Q. E. D.

EXERCISES.

1. All lines joining two points in two parallel straight lines lie in the plane of the parallels.

2. If three parallel or three concurrent lines meet the same straight line, they lie all in the same plane.

3. If two planes are parallel to a given line, their intersection is parallel to the same line.

4. If three lines lie two and two in three planes, they are either concurrent or parallel.

5. If any number of lines diverge from the same point, the planes through any two pairs of these lines intersect in lines through the same point.

6. If any number of lines are parallel, the intersections of the planes through any two pairs of these lines are parallel to them.

7. If a plane intersect a series of planes all passing through a given line, the lines of intersection all pass through the point where the given line meets the plane, or are parallel to the given line, if it does not meet the plane.

8. Planes through a given point and each of a series of parallel lines intersect another plane in lines passing through the point, where that line of the series which passes through the given point meets the plane.

(If the given point is the *eye* and the plane the *picture plane*, the point in the plane is the *vanishing point* of the perspective representation of the parallel lines.)

9. If from the extremities of two parallel lines AB, CD, parallel lines Aa, Bb, Cc, Dd are drawn to meet a plane in a, b, c, d respectively,

$$ab : cd :: AB : CD.$$

10. If in the figure of Th. VII. CD be drawn meeting the plane BGE in K, the figure $BKEG$ is a parallelogram: and the intersection of the diagonals of the parallelogram lies on the line joining the middle points of AD and CF.

11. P, p are two points in the line AB such that $AP = Bp$: through P, p two parallel planes pass meeting a line AC in Q, q, and a line BD in R, r : prove that the triangles PQR, pqr are equal.

12. If parallel lines drawn through the angular points A, B, C, ... of a plane polygon meet a plane parallel to its plane in a, b, c, ... respectively: the polygon abc... is identically equal to ABC.

13. If lines drawn from a point O through the angular points A, B, C... of a plane polygon meet a plane parallel to

its plane in a, b, c... respectively; the polygon abc... is similar to ABC..., and the ratio of corresponding sides is that of Oa to OA.

14. If in the preceding question the plane abc... is not parallel to ABC..., the corresponding sides ab and AB, bc and BC, ... meet in points lying on the intersection of the two planes.

15. If the triangles ABC, abc in different planes are such that AB and ab meet (when produced), as also BC and bc, CA and ca; the lines Aa, Bb, Cc meet in one point, or are parallel.

16. $A_1B_1C_1$, $A_2B_2C_2$ are shadows on the same plane of a triangle ABC cast by two bright points at O_1, O_2: the intersections of B_1C_1 and B_2C_2, of C_1A_1 and C_2A_2, and of A_1B_1, and A_2B_2 lie in one straight line, and A_1A_2, B_1B_2, C_1C_2 meet in one point.

SECTION II.

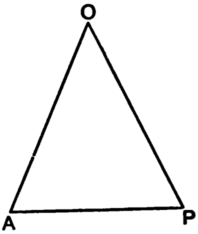
NORMALS AND OBLIQUES.

THEOREM VIII.

There is one point in a plane, and only one, whose distance from a given point without the plane is less than that of any other point, and the straight line drawn from the given point to this point is perpendicular to every straight line through it in the plane.

Let O be a point without a given plane, and P any point in the plane.

Then, since O is not in the plane,
OP cannot be less than a certain finite length.



If A be a point in the plane such that OA is equal to this length, there is no position of P, for which OP is equal to OA:

for if there were, joining *AP* in the plane,

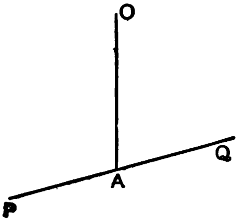
OAP would be an isosceles triangle,

and therefore the angle *OAP* an acute angle:

therefore the perpendicular from *O* on *AP* would be less than *OA*. (Pl. G. I. 15.)

Therefore there is one point *A* in the plane, and only one, whose distance from *O* is less than the distances from *O* of all other points in the plane.

Let *PAQ* be any line, passing through *A*, in the plane,



then *OA*, being the shortest distance from *A* to the plane, is the least distance from *O* to the line *PAQ*;

therefore *OA* is perpendicular to the line *PAQ*, (Pl. G. I. 15.)

so that *OA* is perpendicular to every line in the plane passing through *A*. Q. E. D.

Def. 5. A straight line perpendicular to every straight line that meets it in a given plane is said to be *normal* to the plane, and the plane *normal* to the straight line.

COR. 1. (Euc. XI. 13). From a point outside a plane one, and only one, normal can be drawn to the plane.

COR. 2. Through any point in a straight line one plane, and only one, can be drawn to which the line is normal.

For the plane through *A* together with its normal may be moved till *A* coincides with the given point and the normal *OA* with the given line, which will therefore be normal to the plane in its new position; and no other plane can be drawn through the point, to which the line is normal.

COR. 3. All the perpendiculars to a given line at a given point in it lie in the plane through that point, to which the given line is normal.

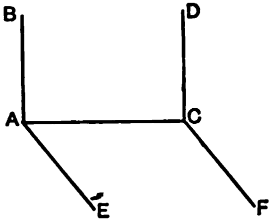
For the plane through *OA* and any line *AM* not in the plane through *A*, normal to *OA*, would cut that plane in a line, *AP* (suppose): and *OAP* being a right angle, *OAM* is not a right angle.

COR. 4. (Euc. XI. 5). If each of three lines, meeting in a point, is perpendicular to the same line, the three lines lie in the same plane.

COR. 5. A line at right angles to a fixed line and revolving about that line describes a plane.

THEOREM IX. (Euc. XI. 8).

If of two parallel lines one is normal to a plane, the other is normal to the same plane.



Let the two parallel lines *BA*, *CD* meet a plane in the points *A*, *C*, and let *AB* be normal to the plane; then *CD* is also normal to the plane.

Take *AE*, *CF*, any two parallel lines in the plane.

Then, because *BA*, *AE* are parallel respectively to *DC*, *CF*, the angle *BAE* is equal to the angle *DCF*: (Th. VII).

but, since *BA* is normal to the plane in which *AE* is, *BAE* is a right angle, and therefore *DCF* is a right angle: and, *CF* being *any* line in the plane, *DC* is normal to the plane. Q. E. D.

COR. 1. (Euc. XI. 13). From a point in a plane one, and only one, normal can be drawn to the plane.

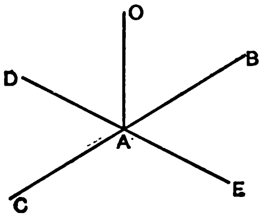
For the parallel, through that point, to a normal from a point outside the plane is itself normal to the plane, and no other line through it can be normal, since in that case a parallel to it through the outside point would be normal, which is impossible, because there is only one normal to a plane from a point outside it. (Th. VIII. Cor. 1.)

COR. 2. (Euc. XI. 6.) Normals to the same plane are parallel.

For if *A*, *C* are points in the plane, there is only one normal through *C* and only one parallel through *C* to the normal through *A*, and (by the Theorem) the parallel is normal; therefore the normal is parallel; or the normals at any two points *A*, *C* are parallel.

THEOREM X. (Euc. xi. 4).

If a line is perpendicular to each of two lines at their point of intersection, it is normal to their plane.



Let the line *OA* be perpendicular to each of the lines *BAC*, *DAE*,

then *OA* is normal to the plane of *BAC*, *DAE*.

For all the perpendiculars to *BC* at *A* lie in one plane, (Th. I. Cor. 3.)

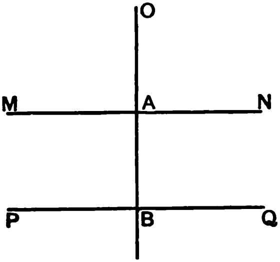
and all the perpendiculars to *DE* at *A* lie in another plane, therefore there is only one line, the intersection of these two planes, which is perpendicular to both *BC* and *DE*, and *OA* is that line.

Also there is only one normal to the plane of *BC*, *DE* at the point *A*, (Th. IX. Cor. 1.)
and that normal is perpendicular to *BC*, *DE*, (Def. 5.)
therefore *OA* is the normal at the point *A*. Q. E. D.

THEOREM XI. (including Euc. XI. 14.)

If two planes are parallel, a normal to the one is also normal to the other : and if the same line is normal to two planes, the planes are parallel.

Let the line *OA*, the normal to one plane at *A*, meet a plane parallel to the first at *B* : then *OAB* is normal to the second plane.



Take *PBQ*, any line through *B* in the second plane, and let the plane through *OAB* and *PQ* intersect the first plane in *MAN*.

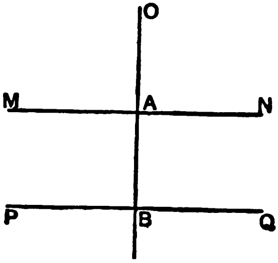
Then, because the two given planes are parallel,
MN is parallel to *PQ*, (Th. IV.)

and OAB is perpendicular to MAN, since OA is the normal to the plane,

therefore OAB is perpendicular to PBQ, any line through B in the second plane,

and therefore OAB is normal to the second plane.

Further, let OAB be normal to two planes through A and B respectively; these planes will be parallel.



For OA, being normal to the plane through A, is also normal to the plane parallel to the first through B (as just proved), and there is only one plane through B normal to OB, (Th. VIII. Cor. 2.)

therefore the plane through B is parallel to that through A.

Q. E. D.

COR. 1. The normals to parallel planes are parallel.

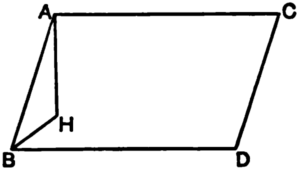
For they are all parallel to any common normal.

COR. 2. If normals to two planes are parallel, the planes are parallel.

For a line parallel to one normal will be parallel to the other, and will therefore be normal to both planes.

THEOREM XII.

The intercepts on parallel lines between two parallel planes are all equal, and the intercepts on the common normals are the least of such intercepts.



Let AB be the intercept on one line, and CD that on a line parallel to it, by two parallel planes; then AB and CD are equal.

Since the two parallel planes are cut by the plane of the parallels AB, CD,
the lines of intersection AC, BD are parallel, (Th. IV.)
therefore $ABDC$ is a parallelogram,
and AB is equal to CD.

Hence, the common normals to the two planes being parallel, the intercepts on the common normals are all equal.

Let AH be the intercept on the normal through A, and join BH.

Then, because AH is normal to both planes,
AH is perpendicular to BH, which is in the plane through B:
therefore AH is less than AB;
that is, any intercept on a common normal is less than any intercept on another line. Q. E. D.

D

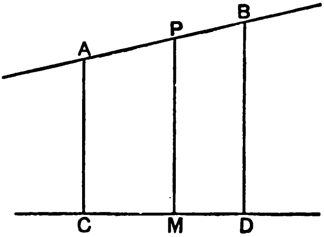
Def. 6. The point where the normal to a plane from a given point without it meets the plane is called the *projection*[1] of the point on the plane.

Def. 7. The projection of a line on a plane is the locus of the projections of all its points.

Def. 8. A line which is neither parallel nor normal to a plane is said to be *oblique* to the plane.

THEOREM XIII.

The projection of a straight line on a plane is the straight line which passes through the projections of any two points in it.



Let A, B be any two points in a given straight line, and C, D their respective projections on a plane: then the straight line through C, D is the projection of the line through A, B.

For AC, BD, being normal to the same plane, are parallel, (Th. IX. Cor. 2.)

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[1] When it is required to distinguish this from other modes of projection, it is termed *orthogonal* projection, but as no other mode of projection is treated of in this book, we may speak of it as projection simply.

and their plane intersects the plane of the projections in the line through *C, D*:

and, *P* being any point in the line *AB* and *M* its projection, *PM* is parallel to *AC* or *BD*, and therefore is in their plane: hence *M* is in the line of intersection *CD*,

so that the line through *C, D* is the locus of the projections of the points in the line through *A, B*, and is therefore the projection of that line. (Def. 6.) Q. E. D.

THEOREM XIV.

If a straight line is drawn from a point in a plane oblique to that plane, the angle which it makes with its projection on the plane is less, and that which it makes with its projection produced is greater, than that which it makes with any other line in the plane: and the angles which it makes with other lines in the plane are less or greater, as they are nearer or farther from the projection. Also it makes equal angles with lines equally inclined to the projection on opposite sides of it, and it is at right angles to that line in the plane which is at right angles to its projection.

From the point *O* in a plane let *OA* be drawn oblique to the plane and let *OB* be its projection on the plane.

With centre *O* and radius *OB* describe a circle in the plane; produce *BO* to *B'*, and take *OP* any radius.

First, the angle *AOB* shall be less than *AOP*, and *AOB'* greater than *AOP*.

For *AB*, being normal to the plane, is at right angles to *BB'* and *BP*, and since *ABP* is a right angle, *AP* is greater than *AB*: (Pl. G. I. 15.)

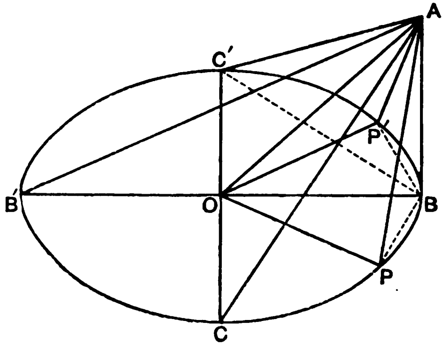
Hence in the triangles *AOB*, *AOP*,

because *AO* is common and *OB* = *OP*, but *AB* less than *AP*,

therefore the angle *AOB* is less than *AOP*. (Pl. G. I. 17).

Also since *ABB′*, *ABP* are right-angled triangles and *BP* is less than *BB′*, if *ABP* were rotated about *AB* into the plane *ABB′*, *P* would come to a point in *BB′* between *B* and *B′*, (Th. VIII. Cor. 5.)

and therefore *AP* is less than *AB′*: (Pl. G. I. 15.)



hence, comparing the triangles *AOP*, *AOB′* (as before *AOP*, *AOB*), (Pl. G. I. 17.)

the angle *AOB′* is greater than *AOP*.

Hence *AOB* is the least, and *AOB′* the greatest, of all the angles which *AO* makes with lines drawn from *O* in the plane.

Also if *OP* is nearer to *OB* than *OC* is, the angle *AOP* is less than *AOC*.

For, the angle BOP being less than BOC, BP is less than BC;

and therefore, if the right-angled triangles ABP, ABC were rotated about AB into the same plane, P would be a point in BC between B and C,

and therefore AP is less than AC;

whence, as before, the angle AOP is less than AOC.

Again on the other side of OB take BOP' equal to BOP, then BP' is equal to BP,

and therefore, from the right-angled triangles ABP', ABP, AP' is equal to AP,

whence, comparing the triangles AOP', AOP, the angle AOP' is equal to AOP,

so that OA makes equal angles with OP, OP', which are equally inclined to its projection OB.

Lastly, if COC' is perpendicular to the projection OB, since the angle BOC is equal to BOC', the angle AOC is equal to AOC',

and they are in the same plane ACC',

therefore AOC, AOC' are right angles,

and OA is at right angles to COC', which is perpendicular to its projection. Q. E. D.

The least of the angles, which an oblique to a plane makes with the lines that meet it in that plane is appropriately termed *the inclination* of the line to the plane. Hence

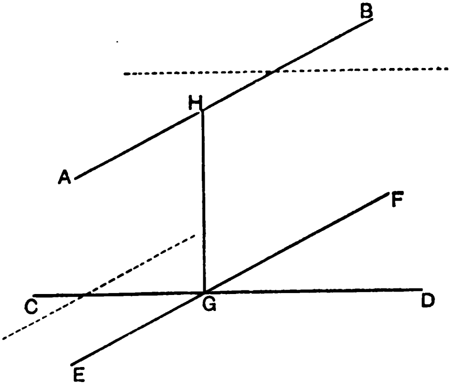
Def. 9. The *inclination of a line to a plane* is the angle which it makes with its projection on the plane.

THEOREM XV.

Of all the straight lines that can be drawn from a point in one straight line to a point in another not in the same plane with the first, there is one and only one which is at right angles to both lines, and its length is less than that of any other of the lines so drawn.

Let *AB*, *CD* be two lines not in the same plane.

Then, if through a point in each a line is drawn parallel to the other, there are two lines which meet parallel respectively to two others which meet, and therefore the plane through the first pair is parallel to that through the other pair. (Th. VI.)



Let *EF* be the projection of *AB* on the plane through *CD*, cutting *CD* in *G*, and let *H* be the point in *AB* of which *G* is the projection:

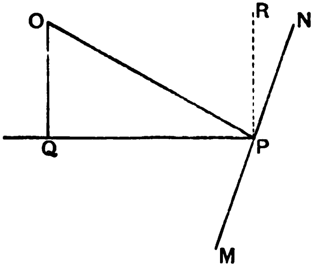
then *HG* is normal to the plane through *CD*, and therefore also normal to the parallel plane through *AB*; (Th. XI.) therefore *HG* is perpendicular both to *AB* and to *CD*.

Also, since *AB* and *CD* are not in the same plane,
no other line joining points in *AB* and *CD* respectively can be parallel to *HG*,
and therefore no line but *HG* can be perpendicular to both *AB* and *CD*.

Further, *HG* being a common normal to two parallel planes, *HG* is less than any other line, not a common normal, joining points in those planes; (Th. XII.)
therefore *HG* is less than any other line joining points in *AB* and *CD*. Q. E. D.

PROBLEM (EUC. XI. 11).

To draw a normal to a given plane from a given point without it.

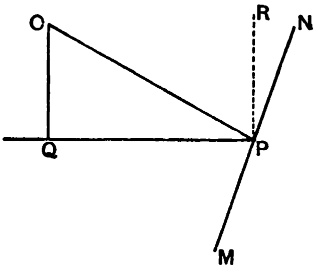


Let *O* be a given point without the given plane.

Draw *OP* perpendicular to *MN*, any straight line in the plane.

Then, if *OP* is not normal to the plane, it is oblique, and if *PQ* is drawn in the plane perpendicular to *MN*, and *OQ* perpendicular to *PQ*, *OQ* is the normal required.

For *PR* (the normal through *P*), *PO*, *PQ*, being all at right angles to *MN*, are in the same plane,
so that *OQ*, *PR* are in the same plane, and, being both perpendicular to *PQ*, are parallel:
therefore *OQ* is normal to the plane. Q. E. F.



COR. (Euc. XI. 12). The normal through a given point in the plane may be drawn by drawing the parallel through that point to a normal drawn from a point without the plane. (Th. IX. Cor. 2.)

Or independently, thus:

Take *MPN*, any line in the plane through the given point *P*, and *OP*, *PQ* perpendicular to *MN*, of which *PQ* is in the plane,
then *PQ* is the projection of *OP*;
and *PR* drawn perpendicular to *PQ* in the plane *OPQ* is the normal to the plane at the point *P*.

EXERCISES.

1. The points in a given plane at a given distance from a point not in the plane lie on the circumference of a circle, and the lines joining such points to the given point are equally inclined to the plane.

2. If the normals drawn from two given points to a plane are equal, the line joining those points is parallel to the plane.

3. If two lines in one plane are equally inclined to another plane, they are equally inclined to the line of intersection of the two planes.

4. Parallel lines are equally inclined to the same plane or to parallel planes.

5. Of the angles contained by any line in one plane and its projection on another plane, the greatest is that made with its projection by the line perpendicular to the intersection of the planes.

6. The range of angles which a given line in one plane can make with lines in another plane is more limited, as the given line is more inclined to the intersection of the planes.

7. O is a point in the intersection of two planes whence OP, OQ are drawn including a given angle, OP in one plane and OQ in the other: show that if the angle POQ be not less than a certain angle, OP or OQ may take any direction in its plane, and determine the limits within which they must lie, if POQ is less than this angle.

8. If from a point O without a plane, OM is drawn normal to the plane and ON perpendicular to the line PQ in the plane, MN is also perpendicular to PQ, and the plane MON normal to PQ.

9. If PM, PN are normals to two planes which intersect in AB, show that AB is normal to the plane MPN.

10. If each of the angles of a quadrilateral is a right angle, the quadrilateral is a rectangle.

11. If two planes are normal to two lines which do not meet and are not parallel, their line of intersection is normal to any plane parallel to both lines.

12. Find a point P in a plane such that the sum of the distances AP, BP from points A, B without the plane is the least possible.

13. If Q be the projection on a given plane of a point P, the projection of PQ on another plane is perpendicular to the intersection of the two planes.

14. A thread is stretched tight from a point A to a point B, passing round a straight edge, not in the same plane with AB, and meeting it at P: show that the parts AP, BP make equal angles with the straight edge, and find the position of P.

15. A thread is stretched tight from A in one face of a pyramid to B in another face, passing over an intermediate face whose edges it meets in P, Q: show how to find the positions of P, Q.

16. The locus of a point equidistant from two given points is a plane.

17. Find the locus of a point equidistant from three given points not in the same straight line.

18. There is one, and only one, point equidistant from four points not all in the same plane.

19. The locus of a point equidistant from two straight lines which meet is a pair of planes perpendicular to the plane of the two lines and to one another.

20. The locus of a point equidistant from three straight lines which meet, but are not in the same plane, is a group of four straight lines.

21. No two lines joining points in two lines which are not in the same plane can be parallel.

22. Through a given point there can be drawn one, and only one, straight line which meets both of two lines which are not in the same plane.

23. A straight line meets three edges of a cube, no two of which are in the same face, in the points P, Q, R; show that, Q being between P and R, PR is least and equal to three times the edge of the cube, when Q is the middle point of one edge and PQR equally inclined to the other two.

24. A line of given length has its extremities on two lines not in the same plane: show that in general for a given position of one end there are two positions of the other, and determine the extreme limits on each line, within which the end on that line must lie.

25. Two fixed straight lines not in the same plane pass through a ring of a given diameter; find the limits between which the ring can play.

SECTION III.

DIHEDRAL AND SOLID ANGLES.

Def. 10. Two planes, which meet and terminate in a straight line, are said to contain, or to make with each other, a *dihedral* angle, and the line in which they meet is termed the *axis* of the angle.

If a plane coincident with one of the planes of a dihedral angle turn about the axis of the angle until it coincides with the other plane, any perpendicular to the axis in the revolving plane will move in a plane perpendicular to the axis, and will turn through a rectilineal angle, whose magnitude will measure[1] the amount of rotation of the plane. Hence

Def. 11. The dihedral angle between two planes is measured by the rectilineal angle between two perpendiculars to the axis from the same point in it, the one in one plane and the other in the other.

Def. 12. When the dihedral angle between two planes is a right angle, the planes are said to be perpendicular to one another.

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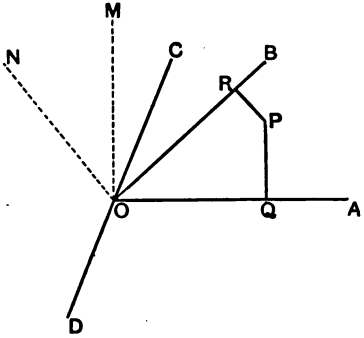
[1] For equal increments of the dihedral angle involve equal increments of the rectilineal angle, which is not the case for any other line than a perpendicular to the axis.

THEOREM XVI.

The dihedral angle between two planes is equal to one, and supplementary to the other, of the angles between normals to the two planes.



Let *CD* be the axis of the dihedral angle between the planes *ACD*, *BCD*,
and let *OA*, *OB* be perpendiculars to *CD* in the two planes,
and let *OM*, *ON* be normals to the two planes drawn from *O*, so that, if the plane *BCD* were made, by revolving round *CD*, to coincide with *ACD*, *ON* would coincide with *OM*.

Then *OM*, *ON*, being normals to the planes, are both perpendicular to *CD*,
and therefore lie in the same plane with *OA*, *OB*.
(Th. VIII. Cor. 3.)

And, since *AOM*, *BON* are right angles, of which the part *BOM* is common, the angle *MON* is equal to *AOB*,

that is, the angle between the normals *OM*, *ON* is equal to the dihedral angle between the planes.

Any other pair of normals to the planes will be parallel respectively to *OM*, *ON*,

and the angles between them will be equal to *MON*, and therefore to the dihedral angle,

if they have both like, or both opposite, senses to *OM*, *ON* respectively;

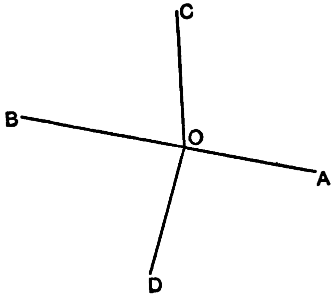
but they will be supplementary to *MON*, and therefore to the dihedral angle,

if one of them has the like, and the other the opposite, sense to *OM*, *ON* respectively. Q. E. D.

COR. If from a point *P*, *within* the dihedral angle, normals *PQ*, *PR* are drawn to its planes, the angle *QPR* is supplementary to the dihedral angle.

THEOREM XVII. (Euc. XI. 18).

Every plane passing through a normal to a given plane is perpendicular to that plane.



Let OC be normal to a given plane, and let a plane through OC cut the given plane in AB, then the plane CAB will be perpendicular to the given plane.

Draw OD in the given plane perpendicular to AB.

Then the angle COD measures the dihedral angle between the plane ACB and the given plane,
but, since OC is normal to the plane,
COD is a right angle,
therefore the plane ACB, passing through OC, is perpendicular to the given plane. (Def. 12.)

Q. E. D.

COR. 1. If a plane is perpendicular to another plane, all perpendiculars to their intersection in the one are normals to the other.

For these perpendiculars, being in the same plane, are parallel, and lines parallel to a normal are normals.

COR. 2. (Euc. XI. 19). The line of intersection of two planes perpendicular to the same plane is normal to that plane.

For the normal at the point, where their intersection meets the plane, lies in each plane.

COR. 3. If three lines are perpendicular, each to both the others, the planes through each pair are also perpendicular, each to both the others : and conversely.

NOTE.—It should be observed that the *direction* of a line is determined definitely, when it is given—

(1) That the line is parallel to a given line ;

(2) That the line is normal to a given plane :

but only indefinitely, when it is given—

(3) That the line is perpendicular to a given line ;

(4) That the line is parallel to a given plane.

Also that, if parallel planes may be said to have the same *direction*, the direction of a plane is determined definitely, when it is given—

(1) That the plane is parallel to a given plane ;

(2) That the plane is normal to a given line :

but only indefinitely, when it is given—

(3) That the plane is perpendicular to a given plane ;

(4) That the plane is parallel to a given line.

Further that two *line-directions* determine one *plane-direction* (of planes parallel to both), and that two *plane-directions* determine one *line-direction* (of lines of intersection of the planes).

Def. 13. When a portion of solid space is separated from the rest by three or more planes, meeting in a point, the planes are said to include a *solid angle;* and the lines of intersection of the planes each with the next are termed its *edges.*

Def. 14. Solid angles contained by three, four, ... or any number of planes are termed respectively *trihedral, tetrahedral,* ... or *polyhedral* angles.

THEOREM XVIII. (Euc. XI. 20).

Any two of the angles between the edges of a trihedral angle are together greater than the third.

This is evident, if the three plane angles are equal; while, if they are unequal, it is only necessary to prove that that one of the angles, which is not less than either of the other two, is less than their sum.

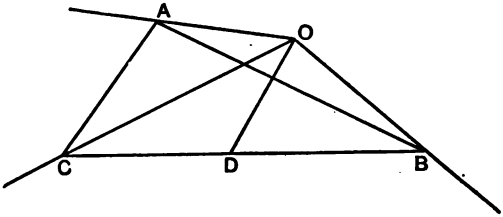
Let O be the vertex of a trihedral angle contained by the three planes BOC, COA, AOB (which may be denoted thus, $O(ABC)$,) and let the angle BOC be not less than AOB and greater than AOC.

In the plane BOC draw OD, making the angle COD equal to COA,

and let any line BC cut it in D;
take OA equal to OD, and join AB, AC.

Then in the triangles DOC, AOC,
$OA = OD$, OC is common to both, and $COA = COD$,
therefore $CA = CD$.

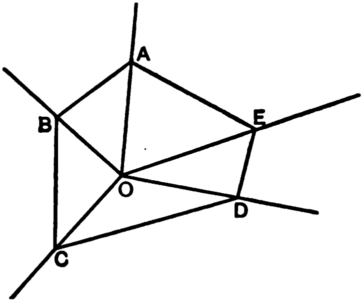
And because BA and AC are together greater than BC or BD and DC,
and $AC = DC$,
therefore AB is greater than BD.



Then in the triangles AOB, DOB,
$OA = OD$, OB is common, and AB is greater than BD,
therefore the angle BOA is greater than BOD:
but the angle $AOC = DOC$,
therefore the two angles BOA, AOC are together greater than BOD and DOC,
that is, than BOC. Q. E. D.

THEOREM XIX.

The sum of the rectilineal angles in the plane faces of a solid angle is less than four right angles.



Let the number of planes which include the solid angle be *n*, let *O* be its vertex and let a plane cut its edges in *A*, *B*, *C*, *D*, ... and its plane faces in the *n* sides of the convex polygon *ABCD* ... : then the sum of the angles in the *n* triangles, whose common vertex is *O*, is 2*n* right angles.

The same would be true, if the point *O* were in the plane *ABC* ..., in which case the vertical angles at *O* would be together equal to four right angles, and the base angles together equal to the angles of the polygon.

But, *O* not being in the plane, there are trihedral angles at each of the points *A*, *B*, *C*, *D* ...

and at *A* the two angles *OAB*, *OAE*, which are two of the base angles of the triangles, are greater than *EAB*, one of the angles of the polygon; (Th. XVIII.)

and similarly at *B*, *C*, *D* ...

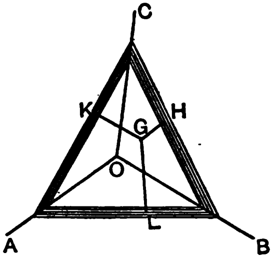
Therefore the base angles of the *n* triangles are together greater than the angles of the polygon,

and therefore, the sum of the base angles and vertical angles being the same as if the vertex *O* were in the plane,

the sum of the rectilineal angles at *O* is less than four right angles. Q. E. D.

THEOREM XX.

The sum of the dihedral angles between the planes including a trihedral angle is greater than two right angles.



From any point *G*, within the trihedral angle *O*(*ABC*), let *GH*, *GK*, *GL* be drawn normal to the planes *BOC*, *COA*, *AOB* respectively.

Then the angle *HGK* is supplementary to the dihedral angle between the planes *BOC*, *COA*, to which *GH*, *GK* are respectively normal, (Th. XVI. Cor.)

so that the angle *HGK* and the dihedral angle whose axis is *OC* are together equal to two right angles.

Similarly for the angles *KGL* and *LGH*.

Therefore the three angles *HGK*, *KGL*, *LGH*, together with the three dihedral angles, whose axes are *OC*, *OA*, *OB*, are equal to six right angles.

But the planes *HGK*, *KGL*, *LGH* include a trihedral angle since they are not in the same plane,
therefore the sum of the angles *HGK*, *KGL*, *LGH* is less than four right angles,
and these together with the three dihedral angles make up six right angles;
therefore the sum of the dihedral angles is greater than two right angles. Q. E. D.

COR. Since *O* is within the solid angle *G*(*HKL*), as the dihedral angles of *O*(*ABC*) and the angles in the plane faces of *G*(*HKL*) are supplementary, so the dihedral angles of *G*(*HKL*) and the angles in the plane faces of *O*(*ABC*) are supplementary.

Ex. If the solid angle be *n*-hedral, the sum of its dihedral angles is greater than $2n-4$ right angles.

THEOREM XXI.

If the angles in two of the plane faces of a trihedral angle are equal, the dihedral angles which these planes make with the third are equal.

Let *O*(*ABC*) be a trihedral angle, and let the angles *AOB*, *AOC* be equal, then the angles which the planes *AOB*, *AOC* make with the plane *BOC* are equal.

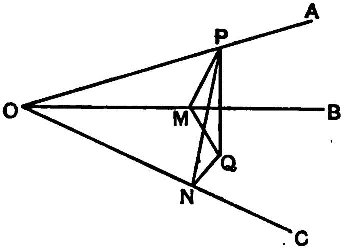
Take any point *P* in *OA*, and draw *PM*, *PN* perpendicular to *OB*, *OC* respectively,

and in the plane *BOC* draw *MQ*, *NQ* perpendicular to the same lines and meeting in *Q*.

Then, since OB is perpendicular to both MP, MQ, it is normal to the plane MPQ: (Th. X.)
so likewise OC is normal to the plane NPQ.

Hence PQ, the intersection of these two planes, is[1] normal to the plane BOC.

Then in the triangles OPM, OPN,
since the angles $POM = PON$, and $PMO = PNO$, and OP is common,
therefore $PM = PN$.



And in the right-angled triangles PQM, PQN,
since $PM = PN$, and PQ is common,
therefore the angle PMQ = the angle PNQ;
that is, the dihedral angle between AOB, BOC is equal to that between AOC, BOC. Q. E. D.

COR. If the three angles in the three plane faces are equal the three dihedral angles are equal, and therefore each is greater than $\frac{2}{3}$ of a right angle.

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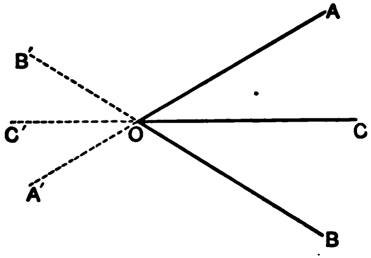
[1] See Ex. 9, Sect. II. Or thus: lines through Q parallel to OB, OC, being normal to the planes MPQ, NPQ, are both perpendicular to PQ, and therefore PQ is normal to their plane, that is, the plane BOC.

Theorem XXII.

A trihedral angle is proportional to the excess of the sum of its dihedral angles over two right angles.



Let $O(ABC)$ be a trihedral angle, and let its dihedral angles, whose axes are OA, OB, OC respectively, be denoted by A, B, C.

If BO be produced to B', the dihedral angle between the planes ABB' and CBB' may be regarded as made up of the two trihedral angles $O(ABC)$ and $O(AB'C)$, so that we may assert that

the two trihedral angles $O(ABC)$, $O(AB'C)$, are together equal to the dihedral angle B,

$$\text{or } O(ABC) + O(AB'C) = B.$$

In like manner, if CO be produced to C,

$$O(ABC) + O(ABC') = C,$$

and if AO be produced to A',

$$O(A'B'C') + O(AB'C') = A.$$

Also, the two dihedral angles between ABB' and CBB' (or B) and between ABB' and $C'BB'$ are together equal to two dihedral right angles,

therefore $O(ABC) + O(AB'C) + O(ABC') + O(AB'C')$ $=$ two right angles:

hence, by taking the sum of the angles equal to A, B, C, it follows that

$O(ABC) + O(A'B'C') +$ two right angles $= A + B + C$; and therefore, assuming that $O(A'B'C') = O(ABC)$ (Section V. Post.)

$O(ABC) = \frac{1}{2}(A + B + C -$ two dihedral right angles).

This relation implies that the unit of measurement of a trihedral angle is the same as that of a dihedral angle; but it follows that, regardless of any particular unit of measurement, $O(ABC)$ is proportional to $A + B + C -$ two right angles. Q. E. D.

COR. The solid angle between three rectangular planes, for which $A = B = C =$ one right angle, is half a dihedral right angle, as is obvious.

Hence the sum of the solid angles about a point including all space is four dihedral right angles.

Ex. An n-hedral angle is proportional to the excess of the sum of its dihedral angles over $2n-4$ right angles.

EXERCISES.

1. If two planes are equally inclined to a third plane, the line of intersection of the two planes is equally inclined to their lines of intersection with the third plane.

2. If the angle AOC is greater than the angle AOB, the dihedral angle between the planes AOB, BOC is greater than that between AOC, BOC.

3. If the angles AOB, AOC in different planes are equal, the plane bisecting the dihedral angle between their planes is perpendicular to the plane BOC.

4. A line drawn within a trihedral angle from its vertex makes with its edges angles, whose sum is less than the sum, and greater than half the sum, of the angles of its plane faces.

5. The sum of the angles subtended by the sides of a triangle at a point not in its plane is greater than that of the angles subtended by the sides of any triangle inscribed in the first.

6. The sum of the angles of a skew (*i.e.* not plane) quadrilateral is less than four right angles.

7. Draw a line equally inclined to three straight lines which meet in a point.

8. If each of the sides of the equilateral triangle ABC subtends a right angle at O, the lines OA, OB, OC are all equal, and the sum of the normals from any point in the plane ABC to the planes BOC, COA, AOB is constant.

9. Draw a plane perpendicular to a given plane so that its intersections with it and another given plane may include a given angle. Examine the limits of possibility of the problem.

10. If a plane revolves about a fixed line, through which it passes, its inclination to a fixed plane varies between a

right angle and a certain minimum angle. Prove this, and trace the variation for different positions of the plane, through a complete revolution.'

11. Find a construction for the trihedral angle, the angles of whose plane faces are equal to three given angles. In what case would the construction fail, and how?

12. If the angles of the plane faces of two trihedral angles are equal, each to each, the trihedral angles are equal, being either *superposable* or *opposable* (*v.* Sect. V. p. 77). Is the same true if the angles are tetrahedral or polyhedral?

13. The locus of a point equidistant from two intersecting planes is the pair of planes which bisect the dihedral angles between them.

14. The locus of a point equidistant from three planes which meet in one point, is the group of four lines, in which the six bisecting planes of the dihedral angles between the planes taken in pairs, intersect three and three.

15. Into how many regions do four planes, no two of which are parallel and no three of which intersect in the same line, divide space?

Show that in each of eight of these regions, and in no others, is there a point equidistant from the four planes.

SECTION IV.

POLYHEDRA AND OTHER SOLIDS.

Def. 15. A finite portion of space bounded on all sides by planes is termed a *polyhedron.* The plane figures which bound it are termed its *faces*, the sides of those figures its *edges*, and the points where the edges meet its *corners* or the vertices of its solid angles.

Def. 16. When a polyhedron is such that no straight line drawn through it meets its bounding planes more than twice, it is said to be *convex.*

In such a case the polyhedron lies entirely on one side of each of its faces.

Def. 17. If the faces of a polyhedron are all equal regular figures, and the same number of faces (or edges) meet in each corner, it is said to be a *regular* polyhedron.

THEOREM XXIII.

The sum of the numbers of faces and corners of any convex polyhedron is greater by two than the number of its edges.

For the polyhedron may be conceived to be built up by starting from one plane face, and adding the others one by one so that one, two, or more *consecutive* sides of each new face coincide with edges of the unfinished solid figure.

Beginning then with one face, the number of edges and corners are equal, being the sides and angles of that face, so that F, C, E denoting the number of faces, corners, and edges respectively,

$$F + C = E + 1.$$

Now if a second face be added with one side coinciding with one of the existing edges, the number of edges added will be *one* less than the number of sides of the face, and the number of corners *two* less than this number, so that the addition to the number of faces and corners together is the same as that to the number of edges, and it still remains true that $F + C = E + 1$.

The same is true for each addition of a face with one side only coinciding with a previously existing edge.

When a face is added with two or more *consecutive* sides coinciding with existing edges, it is easily seen in like manner that the number of edges added is one more than the number of corners added, so that the relation $F + C = E + 1$ still holds.

Hence this relation holds till the addition of the last face to complete the polyhedron, when *no* new edges or corners are added, but *one* new face; therefore, finally,

$$F + C = E + 2.$$

Q. E. D.

COR. Since in this formula F, C are interchangeable, E remaining the same, it suggests that, for every polyhedron, there is another which with the same number of edges has as many faces as the first has corners and as many corners as the first has faces.

This may readily be seen to be true, since, if points be taken in a plane opposite each corner, one point in each of the faces that meet in that corner, the plane faces formed by joining these points will include another polyhedron inscribed in the former with a face for every corner and a corner for every face, the number of edges being the same in both. Such polyhedra have been termed *conjugate* to one another.

NOTE.—The preceding Theorem has been usually ascribed to Euler; but it has recently been shown by M. de Jonquières, in communications to the French Academy of Sciences (February 17 and March 31, 1890), that it was known to Descartes, and employed by him in a posthumously published work entitled *De Solidorum Elementis.* The demonstration above is due to De Morgan, though substantially the same proof, "under mathematical symbols which rather conceal its simplicity," was given by Cauchy. It was extended by Cauchy to the more general case of a convex polyhedron divided into cells, each of which is a convex polyhedron; in which case, if all the faces, corners, and edges of all the polyhedra are counted, and P denote the number of cells, he shows that $F + C = E + P + 1$. This extension, as well as Euler's Theorem, may (as suggested by Listing), by regarding the whole of space as divided into regions, one for each cell and one for the whole external infinite space, be thus expressed: The sum of the numbers of faces and corners is equal to that of the edges and regions. Reckoning corners as points, and so of 0 dimensions, edges of 1 dimension, faces of 2 dimensions, and regions of 3 dimensions, it may be said of any division of space into convex polyhedra and surrounding infinite space, that the number of elements of even dimensions is equal to that of the elements of odd dimensions. Lhuilier obtained a corresponding formula for non-convex polyhedra, and other divisions of space, but the general problem has been most exhaustively treated by J. B. Listing in his classical memoir entitled "*Der Census räumlicher Complexe*," *Abhandlungen der k. Gesellschaft zu Göttingen*, vol. x., 1862.

EXERCISES.

1. No polyhedron can have less than six edges.

2. If the faces of a polyhedron are all triangular, the number of faces is even and four less than twice the number of corners.

3. If the angles of a polyhedron are all trihedral, the number of corners is even and four less than twice the number of faces.

4. If each face has m sides, and n edges meet in each corner,

$$\frac{F}{2n} = \frac{C}{2m} = \frac{E}{mn} = \frac{2}{2(m+n) - mn}.$$

Hence show that the only possible values of m or n are 3, 4, or 5, and that there are only five cases of polyhedra whose faces have all the same number of sides and whose angles are contained all by the same number of planes.

5. Prove that, if F_3, F_4 ... denote the numbers of faces having 3, 4, ... sides, and C_3, C_4 ... the numbers of corners where 3, 4, ... faces meet, so that

$F = F_3 + F_4 + F_5 + \ldots$, and $C = C_3 + C_4 + C_5 + \ldots$,
then $2F = 4 + C_3 + 2C_4 + 3C_5 + \ldots$,
and $2C = 4 + F_3 + 2F_4 + 3F_5 + \ldots$:
and hence the number of faces with an odd number of sides must be even, and the number of corners where an odd number of faces meet also even.

6. From the fact that $2E$ cannot be less than $3F$ or than $3C$, prove that every polyhedron must have some faces, which are either triangular, quadrilateral, or pentagonal, and some corners which are either trihedral, tetrahedral, or pentahedral.

7. The conjugate of a tetrahedron is a tetrahedron.

8. The conjugate of a parallelepiped is an octahedron.

REGULAR POLYHEDRA.

THEOREM XXIV.

There cannot be more than five regular convex polyhedra.

Since there must be at least three faces meeting in each corner, and the sum of the plane angles at each corner is less than four right angles (Th. XIX.), the angles of the regular polygons, forming the faces of a regular polyhedron, must be less than $\frac{4}{3}$ of a right angle. This condition excludes all regular polygons, except pentagons, squares, and equilateral triangles.

The angle of a regular pentagon is $\frac{6}{5}$ of a right angle, and therefore three such angles are less, and four greater, than four right angles : hence there can be only one regular polyhedron, whose faces are pentagonal, and in this the angles are trihedral.

The angle of a square is a right angle, and therefore in like manner there can be only one regular polyhedron whose faces are square, and in this the angles are trihedral.

The angle of an equilateral triangle is $\frac{2}{3}$ of a right angle, and therefore six of these angles make up four right angles: hence there cannot be more than five faces meeting in each corner, but there may be either five, four, or three : so that the angles of a regular polyhedron whose faces are equilateral triangles, may be either trihedral, tetrahedral, or pentahedral.

Thus there may be three regular polyhedra, whose faces are equilateral triangles, one whose faces are squares, and one whose faces are regular pentagons, and there can be no other.

Q. E. D.

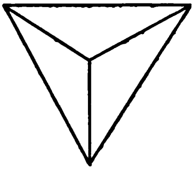
THEOREM XXV.

There are five regular convex polyhedra.

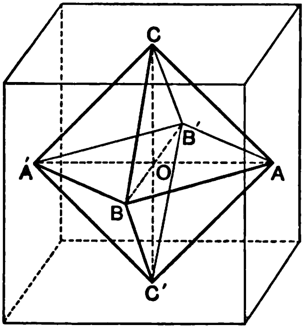
1. Let the vertices of three equilateral triangles be brought together so that their planes include a solid angle, then their bases form another equilateral triangle, which with the other there are the faces of a polyhedron, whose angles are trihedral and equal.

This is the *regular tetrahedron.*

It has four faces, four corners, and six edges.



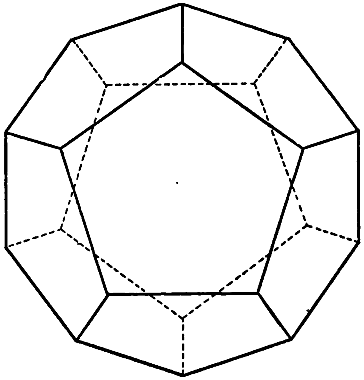
2. Let three equal lines AOA', BOB', COC' be at right angles, each to the other two, and each bisected at their point of intersection O;
then, if planes normal to AOA' be taken through A, A', and similarly planes normal to BOB', COC' through B, B' and C, C' respectively,



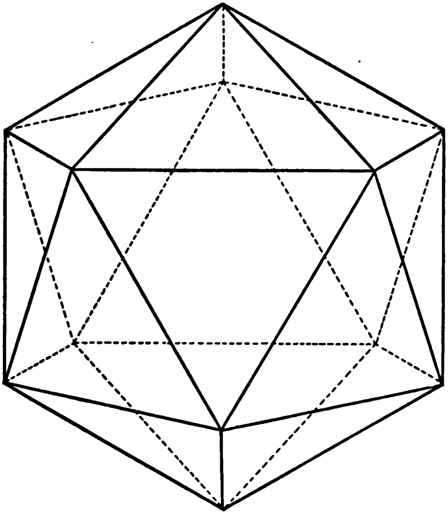
these three pairs of parallel planes include the well-known solid figure, the *cube* or *regular hexahedron*, whose faces are squares and angles trihedral. It has obviously six faces, eight corners, and twelve edges.

If the equilateral triangle ABC be taken, and the like triangles ABC', $AB'C$, &c., in the other seven regions into which the three rectangular planes BOC, COA, AOB divide space, these eight triangles will include a polyhedron, having eight triangular faces, six tetrahedral angles, and twelve edges, which would be inscribed in, and conjugate to, the cube described above. This is the *regular octahedron.*

3. On each side of a regular pentagon let another regular pentagon be described, and let these last be turned about the sides of the first until they meet, then the six pentagons will form an open basket or tray with five completed trihedral angles, five corners with two faces meeting in each, and five corners with a single face: then an exactly similar and equal basket may be placed on the former so that each single-faced corner in the one fits into a double-faced corner of the other, and a polyhedron is completed having twelve pentagonal faces, twenty trihedral angles, and thirty edges. This is the *regular dodecahedron.*



If the centres of the three faces which include one of the trihedral angles be joined, an equilateral triangle will be formed, and the twenty equilateral triangles thus constructed opposite each of the twenty corners will include a polyhedron conjugate to the dodecahedron, with twenty triangular faces, twelve pentahedral angles, and thirty edges. This is the *regular icosahedron.*



EXERCISES.

1. The student should construct the five regular solids by drawing the requisite regular figures on cardboard, each connected with its neighbour by a single edge which should be half cut through; then the whole plane figure thus

drawn being cut out, the faces folded over about the connecting edges will meet and form the several regular polyhedra.

2. The dihedral angles of a regular tetrahedron and octahedron are supplementary.

3. The dihedral angle of a regular tetrahedron is double of the lesser acute angle of the triangle whose sides are 1, $\sqrt{2}$, $\sqrt{3}$, and that of a regular octahedron double of the greater acute angle of the same.

4. Space can be entirely filled up by cubes packed together, but by no other regular polyhedra.

5. The sections of a regular tetrahedron by planes parallel to a pair of opposite edges are rectangles of the same perimeter, and the greatest of such rectangles is a square whose side is half that of one of its edges.

6. A plane is drawn normal to one of the diagonals of a cube through a point in it; examine the nature of its sections with the surface of the cube for different positions of the point. For what section is the area, and the perimeter, a maximum?

7. Planes parallel to the several faces of a regular tetrahedron advance at the same rate, starting from its vertices: trace the nature of the polyhedra into which the tetrahedron is divided at different stages of the advance.

POLYHEDRA, NOT REGULAR, AND OTHER SOLIDS.

It is not the object of the present treatise to develop in detail the properties of particular solid figures; but the definitions of the more important classes of such figures, and a summary of their more fundamental and characteristic properties, will here be given, as likely to be useful to the student. The proofs of these properties will not in general be found difficult by any one who has mastered the previous sections and the standard propositions of plane geometry.

Def. A *parallelepiped* is a solid figure enclosed by three pairs of parallel planes.

It has obviously six faces, eight corners, and twelve edges, and its angles are trihedral.

The following properties being easily proved, may be left as exercises for the student:

1. The opposite faces of a parallelepiped are identically equal parallelograms.

2. The three diagonals joining opposite corners meet in a point and are all bisected in that point.

3. The dihedral angles between faces which intersect in parallel edges are equal or supplementary, according as the edges are opposite or in the same face.

4. In general, no two trihedral angles can be made to coincide by superposition, although opposite angles have the angles in the bounding planes equal and the dihedral angles between them also equal. What exceptional cases are there?

5. Examine the sections made by a plane parallel to that through the extremities of three edges, which meet in one of the corners, and determine the greatest of such sections.

PRISM. *Def.* If two identically equal polygons have their corresponding sides parallel, these with the planes through the corresponding sides include a solid figure termed a *prism*. It is plain that the faces through the parallel sides are parallelograms, and the edges in which these faces meet are all parallel and equal. Any one of these edges may be called the *length* of the prism.

According as the polygons are triangular, quadrilateral, &c., the prism is said to be triangular, quadrilateral, &c.

A *parallelepided* is obviously a *parallelogrammic* prism.

If the parallel edges are normal to the planes of the polygon, the prism is said to be a *right* prism : otherwise, an *oblique* prism.

If the polygon has n sides, the prism has $\overline{n+2}$ faces, $2n$ corners, and $3n$ edges, and its solid angles are trihedral.

It is easily proved that,

1. Parallel plane sections, by planes not meeting the end faces, are identically equal plane polygons.

2. Such sections are least in area when the planes are normal to the parallel edges.

3. The sections by planes parallel to the length of the prism are parallelograms.

CYLINDER. If a line move parallel to a fixed line always passing through a given curve, it describes a *cylindrical* surface, and the solid included between two parallel plane sections (not parallel to the fixed line) and the surface is termed a *cylinder*.

If the guiding curve is a circle, it is called a *circular* cylinder : and if the generating line is normal to the plane of the circle, it is said to be a *right*, otherwise an *oblique*, circular cylinder.

A cylinder may be regarded as the limiting case of a prism, when the number of sides of the base is increased without limit, its perimeter remaining finite.

The properties enunciated above for a prism are true also for a cylinder, *mutatis mutandis.*

PYRAMID. *Def.* A *pyramid* is the solid figure included between a solid angle and a plane which intersects all its bounding planes; and the polygon contained by the lines of intersection is termed its *base.*

It is evident that all the faces except the base are triangular.

If the solid angle is trihedral, the base is triangular; if tetrahedral, the base is quadrilateral; and so on.

A prism may be regarded as the limiting case of a pyramid when the vertex is removed to infinite distance from the base.

A pyramid on a triangular base is a tetrahedron.

If the base have n sides and therefore the vertical angle is n-hedral, the pyramid has $\overline{n+1}$ faces, $\overline{n+1}$ corners, and $2n$ edges, and the other solid angles are trihedral.

The following general properties of a pyramid are easily proved:

1. The sections of a pyramid by parallel planes are similar polygons, having their corresponding sides in the same plane through the vertex and in the same ratio as the distances of the planes from the vertex measured along any one of the edges.

2. If A, B, C, D .. and a, b, c, d ... be the points where two intersecting planes meet the edges through the vertex, then AB and ab, BC and bc, &c., meet respectively in points which lie on the line of intersection of the planes.

3. If $ABCD$... and $abcd$... are two plane polygons, so situated that AB and ab, BC and bc, &c., meet in the same straight line, the lines, Aa, Bb, Cc ... all meet in a point, the vertex of a pyramid of which the polygons are plane sections.

CONE. *Def.* If a line passing through a fixed point moves so as to always pass through a given curve, it describes a *conical* surface, and the solid included between this surface and a plane not passing through the fixed point (or *vertex*) is termed a *cone.* The plane boundary of the cone is termed its *base.*

A cone on a circular base is said to be *right* or *oblique,* according as the line joining the vertex to the centre of the base (termed the *axis*) is normal or oblique to the plane of the base.

A cone may be regarded as the limiting case of a pyramid, when the number of sides of the base is increased without limit, the perimeter remaining finite. Also a cylinder as the limiting case of a cone when the vertex is moved to infinity.

The properties enunciated above for the pyramid are true, *mutatis mutandis*, for the cone.

The properties of the plane sections of the circular cone and cylinder form a large branch of geometry, treated under the head of Conic Sections.

SPHERE. *Def.* The surface-locus of a point, whose distance from a fixed point is equal to a given distance, is a *spherical surface*, and the solid of which it is the bounding surface a *sphere*. The fixed point is the *centre*, any line from the centre to the surface a *radius*, and any line through the centre terminated both ways by the surface a *diameter*.

It follows from the definition that,

1. Two spheres which have the same centre and equal radii coincide completely, and however one of them is moved about the centre fixed, it continues to coincide with the other.

2. The section of a sphere by any plane through its centre is a circle of the same radius as the sphere. Hence a sphere may be regarded as the solid generated by the revolution of a circle about one of its diameters.

3. The section of a sphere by a plane not passing through its centre is a circle, whose radius is less than that of the sphere, and less as the distance of the plane from the centre is greater. (Cf. Ex. 1. Sect. II.)

Def. The circle, in which a plane through the centre cuts a sphere, is termed a *great circle* of the sphere, and the extremities of the diameter normal to the plane the *poles* of the great circle.

4. A great circle can be drawn through any two points on a spherical surface; and *only one*, unless the points are the extremities of a diameter, in which case every great circle through the one passes through the other.

5. The arc of a great circle from any point on a given great circle to either of its poles is a *quadrant* or fourth part of the circumference of a great circle.

6. A sphere is a perfectly *symmetrical* figure. (Cf. Appendix, Sect. V.)

It has *point-symmetry* or *central-symmetry* with respect to the centre, *axial-symmetry* with respect to every diameter, and *plane-symmetry* with respect to every plane through its centre.

7. The tangent lines drawn at a given point on a spherical surface to all the great circles which pass through that point lie in a plane, which is called the *tangent plane* to the surface at the given point, and which is normal to the radius to the point.

8. One, and only one sphere, can be drawn through four points not in the same plane.

TETRAHEDRON. As among plane figures the triangle is the simplest as having the least possible number of sides, so among solid figures the tetrahedron is the simplest as having the least possible number of faces, and thus has many properties analogous to those of the triangle, the proofs of which are in many cases suggested by those of the analogous properties of the triangle. Among these the following are noteworthy:

1. Any three of the faces of a tetrahedron are greater than the fourth.

2. The six planes through each edge and the middle point of the opposite edge intersect in four lines, one through each corner: these lines are the lines joining the vertices with the mean points (or centroids) of the opposite faces, and they meet in one point (the mean point or centroid of the tetrahedron), which divides each of the lines in the ratio of 3 to 1.

3. The three lines joining the middle points of opposite edges of a tetrahedron meet in its mean point, and are bisected in it.

4. The six planes which bisect the dihedral angles of a tetrahedron intersect in four lines, one through each corner, and these meet in one point, the centre of the inscribed sphere.

5. There are seven other spheres, which touch the four planes of the faces outside the tetrahedron. (Cf. Ex. 13, Sect. III.)

6. If the sum of each pair of opposite edges is the same, a sphere may be described touching all the edges.

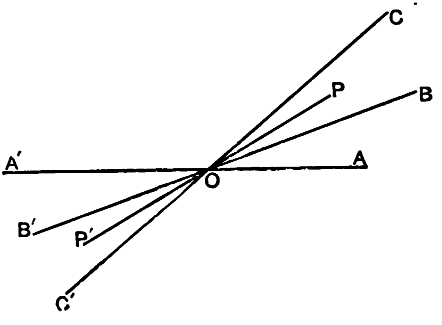
7. The six planes normal to the several edges through their middle points meet in four lines normal respectively to the four faces through the centres of the circumscribing circles of the triangles: and these four lines meet in one point, the centre of the circumscribing sphere.

SECTION V.

VOLUMES OF SOLIDS. STEREOMETRY.

Solid angles, or polyhedra, which have each face of the one identically equal to a corresponding face of the other and the dihedral angles between corresponding faces equal are not *necessarily superposable*, that is, capable of being placed so as to coincide completely.

Thus, if AOA', BOB', COC', are three straight lines passing through O,



the angles $B'OC'$, $C'OA'$, $A'OB'$ are respectively equal to BOC, COA, AOB, and the dihedral angles between the planes of the one set equal respectively to the corresponding angles between those of the other, but the solid angle $O(A'B'C')$ is not superposable on $O(ABC)$.

For if OA' be brought to coincide with OA, and OB' with OB, OC' and OC must lie on opposite sides of the plane AOB: and if OA' be made to coincide with OB and OB' with OA, though OC and OC' will then lie on the same side of the plane AOB, they will not coincide, except in the case where the angles COA, COB are equal. Yet for every point P within the one angle, there is a corresponding point P', in PO produced so that $OP' = OP$, which is related to the edges and planes of $O(A'B'C')$ in identically the same manner in which P is related to those of $O(ABC)$, and hence it seems natural to assume that these solid angles are equal. Similar remarks may be made with respect to the pyramids cut off from these solid angles by two parallel planes, but a fuller discussion of the whole matter is reserved for a later page. (See pp. 98 to 108.)

The solid angles $O(ABC)$, $O(A'B'C')$ may be said to be *opposed*, and solid angles, which can be placed so as to be opposed, may be termed *opposable*.

Generally, if two polyhedra are such that they can be placed so that, every corner of the one being joined with a corresponding corner of the other, the lines so joining them meet and are bisected all in the same point, the polyhedra, not being in general superposable, are said to be *opposable*; and we shall make the following postulate:

POST[1] *Opposable solid angles, or polyhedra, are equal.*

---

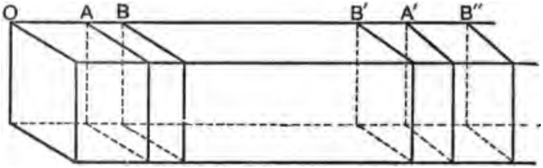
[1] Euclid tacitly assumes this in the definition (XI. Def. 10) "Equal and similar solid figures are such as are contained by similar plane faces equal in number and magnitude," which he directly applies in XI. 28 (see Th. XXIX.). Modern editors, following Simson, have introduced Propositions B., C. proved for *superposable* figures (at least that is what, I suppose,

Def. The normal distance between any face of a parallelepiped, taken as its *base*, and the face parallel to it is termed its *height*.

Def. A rectangular parallelepiped, that is, one whose faces are rectangles and therefore all its angles and dihedral angles right angles, is termed a *cuboid*.[1]

THEOREM XXVI.

Cuboids and right prisms on the same base are to one another as their heights.



Let OA, OB be coincident edges of two cuboids on the same base, then the volumes of the cuboids are in the ratio of OA to OB.

---

they must be understood to mean by adding to "similar" the words "alike situated"); and then in XI. 28 amend (!) Euclid by applying Prop. C. to a non-superposable case.

[1] The need of some short word in the place of the polysyllabic "rectangular parallelepiped" has been long felt. I have coined the word "cuboid" in the hope that it may obtain currency among mathematicians. De Morgan's term "right solid" has *not*, I think, obtained currency.

Let OA and the edges parallel to OA be indefinitely produced,

and in OA thus produced let A' be taken so that $OA' = m \,.\, OA$, where m is any whole number,

then two points in the same line B', B'' can be found such that, while $OB' = n \,.\, OB$ and $OB'' = \overline{n+1} \,.\, OB$, OB' is less, and OB'' greater, than OA, or else $OB' = OA'$.

But the volume of the cuboid on OA' is m times that of the cuboid on OA,

since cuboids on the same base and of equal heights are superposable and therefore equal :-

and so also the volumes of the cuboids on OB', OB'' are respectively n and $\overline{n+1}$ times that of the cuboid on OB.

Also it is evident that the cuboid on OB' is less, and that on OB'' greater, than the cuboid on OA', or else the cuboid on OB' is equal to that on OA',

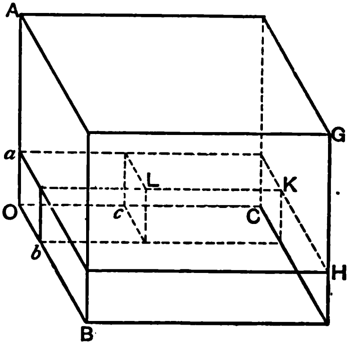
therefore, by the definition of equal ratios,

the cuboid on OA is to that on OB as OA to OB.

It is plain, that since the above reasoning is independent of the form of the base, the same conclusion is true for any other right prisms. Q. E. D.

THEOREM XXVII.

The volumes of cuboids have to one another the ratio compounded of the ratios of the three edges of any one of their solid angles.



Let two cuboids OG, OL be placed so that two of their solid angles coincide at O, and that the edges OA, OB, OC of the one coincide in direction with the edges Oa, Ob, Oc of the other respectively, then the volume of OG is to that of OL in the ratio compounded of the ratios, OA to Oa, OB to Ob, OC to Oc.

Let the plane through a parallel to OBC cut the edge through G parallel to OA in H,

then, OG, OH being cuboids on the same base and OA, oa their heights,

vol. of OG : vol. of OH :: OA : Oa. (Th. XXVI.)

Next let the plane through b parallel to OAC cut the edge through H parallel to OB in K,

then for a like reason

vol. of OH : vol. of OK : : OB : Ob.

Lastly, let the plane through c parallel to OAB cut the edge through K parallel to OC in L,

then vol. of OK : vol. of OL : : OC : Oc.

Hence, compounding these three proportions,

vol. of OG : vol. of OL = the ratio compounded of the ratios $OA : Oa$, $OB : ob$, $OC : Oc$. Q. E. D.

COR. 1. If the edges OC, Oc of the cuboids are equal, the cuboids are to one another in the ratio compounded of the ratios of their other edges, OA to Oa and OB to Ob, and therefore in the ratio of their bases CB to Cb. Hence, cuboids of the same height are to one another as their bases.

COR. 2. If the cuboid OL is a cube, each of whose edges is the unit of length (denoted[1] by L), which is usually taken as the unit of volume and termed a cubic unit (denoted by LLL or L^3), and a, b, c denote respectively the ratios of OA, OB, OC to the unit of length L, the volume of the cuboid $OG = abc$ LLL.

---

[1] It is highly desirable that this expressive notation for units founded on the unit of length should be universally adapted, and that the student from his first introduction to the measurement of areas and volumes, writing F for a foot, should write FF or F^2 for a square foot and FFF or F^3 for a cubic foot. It adapts itself readily to cuboidal units, not cubes, thus I denoting an inch, FI denotes an area a foot long and an inch broad, and FII a cuboid with its base a square inch and height a foot. With this notation the rule of "duodecimals" explains itself.

If the rectangle *AB* be the base, its area $= ab$ LL, therefore, writing *A* for the product *ab*, the volume of the cuboid $= Ac$ LLL.

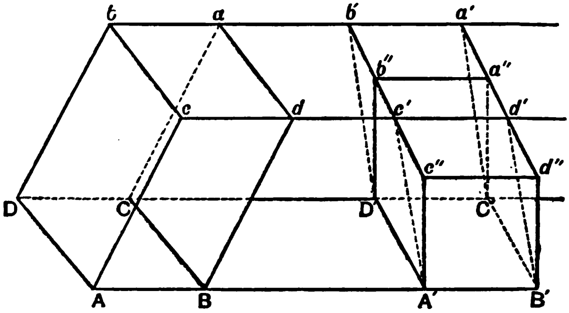
These results are usually expressed thus:

The volume of a cuboid (in cubic units) = the product of the lengths (in linear units) of three conterminous edges and also = the product of its base (in square units) and its height (in linear units).

COR. 3. The like theorem is true for equiangular parallelepipeds, that is, such as have the angles of the faces in the one equal respectively to those in the other, and therefore the dihedral and solid angles also equal.

THEOREM XXVIII.

The volume of a parallelepiped is equal to that of a cuboid whose base and height are equal respectively to those of the parallelepiped.



Let Aa be any parallelepiped and $ABCD$ its base.

Let the edges parallel to AB be indefinitely produced, and, A' being any point in AB, take $A'B'$ equal to AB in the same line;
through A', B' draw two planes normal to $ABA'B'$,
then, these with the planes of Aa through the edges that have been produced will include a right parallelepiped $A'a'$.

Now if the solid included between the planes Acb, $A'c'b'$ is moved so that its corners move along AB and the edges parallel to AB until the plane $AcbD$ coincides with $BdaC$, the plane $A'c'b'D'$ will finally coincide with $B'd'a'C'$, since $A'B' = AB$,
therefore the solid between $AcbD$ and $A'c'b'D'$ is equal to that between $BdaC$ and $B'd'a'C'$,
and, taking from these the part common to both,
there remains the parallelepiped Aa equal to the right parallelepiped $A'a'$.

Next through $A'B'$, $C'D'$ take two planes normal to $A'D'$ or $B'C'$,
then there will be a cuboid $A'a''$ with the base $A'B'C'D'$, and the height Ac'',
and since the solid wedge between the planes $D'a''$ and $D'a'$ may be moved and brought to coincide with that between $A'd''$ and $A'd'$,
the parallelepiped $A'a'$ is equal to the cuboid $A'a''$.

Hence the parallelepiped Aa is equal to the cuboid $A'a''$, and the base $A'B'C'D'$ of the cuboid being equal to $ABCD$,

(since $A'B'$ is equal to AB and they are between the same parallels,)
and $A'c''$, the height of the cuboid, being also the height of the parallelepiped, the parallelepiped Aa is equal to a cuboid, whose base and height are respectively equal to the base and height of the parallelepiped. Q. E. D.

COR. 1. Parallelepipeds on equal bases and of equal heights are equal in volume.

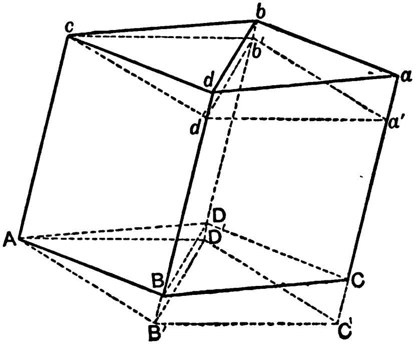
For they are equal to the same cuboid.

COR. 2. If the base of a parallelepiped = ALL, and its height hL, its volume = AhLLL. (Th. XXVII. Cor. 2.)

Or, the volume of a parallelepiped (in cubic units) is the product of its base (in square units) and its height (in linear units.)

THEOREM XXIX.

A parallelepiped is bisected by the plane through any two parallel diagonals of opposite faces.



Let BD, bd be two parallel diagonals of the opposite faces AC, ac of the parallelepiped Aa,

then the plane $BDbd$ will divide it into two equal prisms.

The triangular prisms Abd and aBD are equal, edge for edge, face for face, and dihedral angle for dihedral angle,

but they are not superposable, if Ac is oblique to the plane ABC.

They are however opposable,

for the prism aBD may be so placed that a coincides with A, and aC, ad, ab are respectively in the prolongations of the lines CA, DA, BA,

therefore, by the postulate that opposable solids are equal, the prisms are equal.

Q. E. D.

ALTERNATIVE PROOF (not involving the postulate of "opposable" figures).

Let the sections of the parallelepiped by planes through A and c normal to Ac be $AB'C'D'$, $cd'a'b'$,

then the solid figure between the parallelogrammic faces ca, ca' could evidently be placed so that the face ca should coincide with AC and ca' with AC',

therefore the oblique parallelepiped Aa is equal to the right parallelepiped Aa',

and the prisms Abd, aBD respectively equal to the prisms $Ab'd'$, $a'B'D'$:

but the prism $a'B'D'$ is superposable on $Ab'd'$ by making $a'C'$ coincide with CA and turning the prism about CA until

$a'd'$ coincides with cb', as it can, since they are in the plane through c normal to CA,

therefore the prism $a'B'D'$ is equal to $Ab'd'$,

and therefore the prism Abd is equal to aBD. Q. E. D.

THEOREM XXX.

The volume of a prism is that of a cuboid on an equal base and of equal height.

Every triangular prism is half of a parallelepiped of the same height, whose base is the parallelogram contained by two of the edges of the base of the prism, and is therefore double of that base, as is obvious from the figure of the last theorem : therefore its volume is *half* that of a cuboid of the same height, whose base is *double* of the base of the prism, (Th. XXVIII.)
and therefore equal to that of a cuboid of the same height, whose base is equal to the base of the prism.

If the base of the prism is polygonal, the opposite polygonal faces may be divided similarly into triangles, and the planes through each pair of corresponding sides will divide the prism into triangular prisms, each of which is equal to a cuboid of the same height on an equal base, and, since the sum of such cuboids can be made into a single cuboid whose base is the sum of their bases and therefore equal to the polygonal base of the prism,

the volume of the prism is that of a cuboid on an equal base and of the same height. Q. E. D.

COR. 1. Prisms on equal bases and of equal heights are equal in volume.

COR. 2. If the base of a prism $= A\mathsf{LL}$ and its height $= h\mathsf{L}$, the volume of the prism $= Ah\mathsf{LLL}$:

or the volume of a prism (in cubic units) is the product of its base (in square units) and its height (in linear units).

COR. 3. If planes be taken through any number of generating lines of a cylinder in successive order, they will with the end planes form an inscribed prism, of which the cylinder itself is the limit, when the number of sides is increased without limit, while the curved bases of the cylinder are the limits of the polygonal bases of the prism. Hence we infer that,

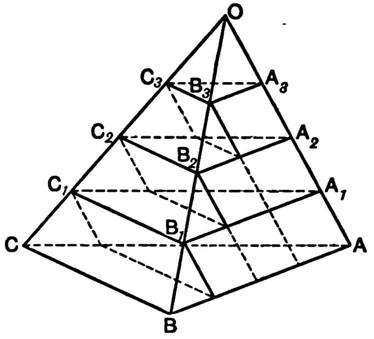
The volume of a cylinder (in cubic units) is the product of its base (in square units) and its height (in linear units).

Ex. Show that the volume of an oblique prism is equal to that of a right prism with the same length of edge, and whose base is the projection of its base on a plane normal to the edge: and hence if from any point A the normal AB be drawn to the plane of a polygon, and AC normal to another plane and meeting the plane of the polygon in C,

the area of the polygon : the area of its projection on the other plane :: $AC : AB$.

THEOREM XXXI.

Triangular pyramids on equal bases and of equal heights are equal in volume.



Let $OABC$ be a triangular pyramid, and let the edge OA be divided into any number of equal parts in the points A_1, A_2, A_3, ... and let plane sections parallel to the base ABC be taken through each of these points, $A_1B_1C_1$, $A_2B_2C_2$, &c.: through each of the bases of the sections in the plane OBC, B_1C_1, B_2C_2, ... let planes parallel to OA be taken and extending each to the plane next in succession below it.

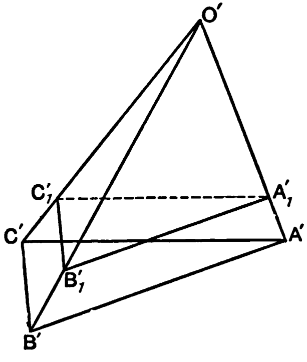
Then the solid figure between each consecutive pair of plane sections is made up of a prism and a wedge,

and it is plain that the bases of all the wedges, parallel to ABC, if brought down to the plane ABC, each corner along a line parallel to OA, would (with the base of the topmost pyramid) just fill up the base ABC, while the edges parallel to the bases would lie in the plane $A_1B_1C_1$,

so that the sum of all the wedges would be less than the solid between the base ABC and the first plane section $A_1B_1C_1$:
hence the pyramid is greater than the sum of the prisms by less than the frustrum between the base and the first plane section.

But by dividing OA into a sufficiently large number of parts, each part may be made as small as we please, and therefore the height of this lowest frustrum and the frustrum itself as small as we please, so that the pyramid may be made to differ from the sum of the prisms by as small a quantity as we please: hence the pyramid is the limit of the sum of the prisms, when their number is increased, and consequently the height of each diminished, without limit.

Now let a second pyramid on an equal base and of equal height be divided similarly to the former by planes parallel to its base; then in the first pyramid any section, as $A_1B_1C_1$. being similar to the base ABC, the area of the section $A_1B_1C_1$: that of the base in the duplicate ratio of A_1B_1 to AB and therefore of OA_1 to OA;



so also in the second pyramid, if $O'A'$ be an edge similarly divided in A'_1, A'_2, ... so that $OA'_1 : OA' :: OA_1 : OA$, &c., the area of the section $A'_1B'_1C'_1$ is to that of the base in the duplicate ratio of OA'_1 to OA' and therefore of OA_1 to OA:

hence, the bases being equal, the corresponding sections are equal.

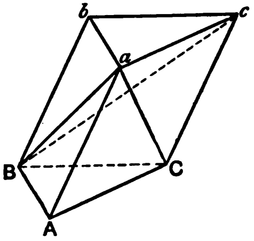
Therefore in the two pyramids there are the same number of prisms, each in the one having its base and height equal to those of the corresponding prism in the other, so that the sum of the prisms in the one is equal to that of the prisms in the other.

Hence, when the number of prisms is increased without limit, the limit of their sum in the one is equal to that of their sum in the other, and therefore the one pyramid is equal to the other. Q. E. D.

COR. If the base of a pyramid is a polygon, it may be divided into triangles to which equal triangles may be constructed so as together to make a single triangle equal in area to the polygon; then, the original pyramid being divided into triangular pyramids by planes through the vertex and the sides of the triangles, these pyramids are separately equal to pyramids of the same height on the triangles which make up the single triangle, and therefore the sums of these pyramids are equal, so that the pyramid on a polygonal base is equal to a pyramid on an equal triangular base of the same height. Hence the proposition may be extended to *any* pyramids on equal bases of equal height.

THEOREM XXXII.

The volume of a triangular pyramid is one-third of that of a prism on the same base and of the same height.



Let Abc be a triangular prism and $aABC$ a triangular pyramid on the same base ABC and of the same height, then the pyramid is one-third of the prism.

Join Bc.

Then the planes aBC and Bac divide the prism into three pyramids $aABC$, $Babc$, $BacC$:
but $aABC$ and $Babc$ are equal, for they are on equal bases ABC and abc and of the same height, (Th. XXXI.)
also $aABC$ and $BacC$ are equal, for they are on equal bases ACa and Cca and of the same height, (Th. XXXI.)
therefore the three pyramids are all equal, and each is one-third of the prism. Q. E. D.

COR. 1. The same theorem is true for a pyramid on a polygonal base. For if the base is divided into triangles by lines drawn from a point within it, the planes through the vertex and these lines will divide the pyramid and prism into the same number of triangular pyramids and prisms, and the theorem, being true for each triangular pyramid and prism on the same base, is true for the pyramid and the prism, which are the sums of each.

COR. 2. If the base of the pyramid = ALL, and its height = hL, its volume = $\frac{1}{3}Ah$LLL :
or, the volume of a pyramid (in cubic units) is one-third of the product of its base (in square units) and its height (in linear units.)

COR. 3. Pyramids on equal bases are to one another as their heights, and pyramids of equal heights are to one another as their bases.

COR. 4. The volume of a tetrahedron (as $BCca$) is one-third of that of a prism, whose base is a triangle (as BCA) with two of its sides equal and parallel to two opposite edges (BC, ca) and height the perpendicular distance between those edges.

Hence if the lengths of two opposite edges be a, bL respectively, the perpendicular distance between them hL, and the angle between them α, the volume of the tetrahedron = $\frac{1}{6}abh \sin \alpha$LLL.

COR. 5. If a polygon be inscribed in the base of a cone, the pyramid contained between planes through its vertex and the sides of the polygon and the polygon as base will be a pyramid inscribed in the cone, and as the number of sides is increased without limit and their lengths correspondingly diminished without limit, the polygon ultimately becomes the curve base of the cone and the inscribed pyramid the cone itself. Hence,

The volume of a cone (in cubic units) is one-third of the product of the area of its base (in square units) and its height (in linear units.)

Ex. 1. The volume of a tetrahedron is two-thirds of that of a parallelepiped, whose base is equal to the section of the tetrahedron by a plane parallel to two opposite edges midway between them, and whose height is the distance between the same edges.

Ex. 2. If two lengths are measured on two fixed lines, the tetrahedron of which these two lengths are opposite edges is of constant volume in whatever parts of the lines the lengths are measured.

THEOREM XXXIII.

A frustrum of a triangular pyramid may be divided into three pyramids, one of which is the mean proportional between the other two.

Let $ABCabc$ be a frustrum of a pyramid between the two parallel planes ABC, abc;

then the planes aBC, cBC divide it into the three pyramids $aABC$, $Babc$, $BacC$,

of which the last is the mean proportional between the other two.

Join Ac meeting aC in n, and bC meeting Bc in m.

Then $ABaC$, $cBaC$ being pyramids on the same base

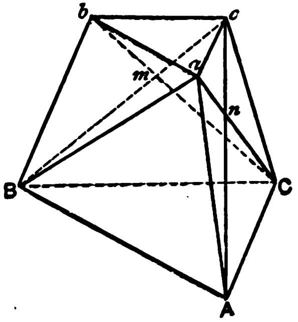
BaC are in the ratio of their heights and therefore of *An* to *nc* and therefore of *AC* to *ac* :

also *CBac*, *bBac* being pyramids on the same base *Bac* are in the ratio of their heights, and therefore of *Cm* to *mb* and therefore of *CB* to *cb* or of *AC* to *ac*:

hence the vol. of *aABC*: that of *BaCc*: : that of *BaCc* : that of *Babc*;

or *BaCc* is the mean proportional between *aABC* and *Babc*.

Q. E. D.



COR. If the bases *ABC*, *abc* respectively $= A_1, A_2$ LL, and the height of the frustrum is hL,

the volumes of the pyramids *aABC*, *Babc* contain, respectively, $\frac{1}{3}A_1h$, $\frac{1}{3}A_2h$ LLL,

and the volume of $BacC = \frac{1}{3}\sqrt{A_1A_2} \cdot h$ LLL,

therefore the volume of the frustrum =

$$\tfrac{1}{3}(A_1 + \sqrt{A_1A_2} + A_2)h \text{ LLL.}$$

Ex. 1. From this theorem deduce Th. XXXII.

Ex. 2. The tetrahedra whose opposite edges are any pair of sides in the two bases, which are not parallel, are all equal.

Ex. 3. In how many different ways can the frustrum be divided as in the Theorem?

SIMILAR SOLIDS.

Through all the corners of any polyhedron, *e.g.* the tetrahedron $ABCD$, let straight lines be drawn meeting in the same point O, and let a, b, c, d, be points in the lines through A, B, C, D respectively,

such that $Oa : OA :: Ob : OB :: Oc : OC :: Od : OD$,

then it easily follows (from *Plane Geom.* and Th. VI.) that the polyhedron $abcd$ has its edges and faces parallel each to a corresponding edge or face of $ABCD$, and consequently corresponding angles in corresponding faces equal, corresponding dihedral angles equal, and the ratios of corresponding edges all equal. This will be true, whether a, b, c, d lie on the same side of O as A, B, C, D or on the opposite side, but in the former case, if a were brought to A, the solid angle $a(bcd)$ might be brought into coincidence with $A(BCD)$, while in the latter $a(bcd)$ could be opposed to, but not made coincident with $A(BCD)$.

In either case the polyhedron $abcd$ is said to be similar to $ABCD$, and the two cases may be distinguished as *directly* similar and *oppositely* similar.

Hence we are led to the following definition:

Def. Similar polyhedra are such as have each face of the one similar to a corresponding face of the other, and similarly placed with respect to adjoining faces.

From this definition the converse of what has been stated above follows, namely, that similar solids can be so placed with respect to each other that their corresponding edges and faces are all parallel, and then the lines joining the corresponding corners meet in one point, which is on the same side of corresponding faces if the polyhedra are directly similar, but between corresponding faces if they are oppositely similar. (Cf. *Plane Geom.* V. 6.)

THEOREM XXXIV.

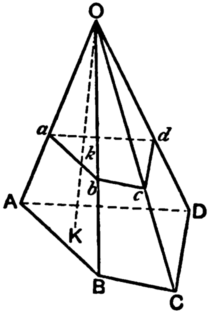
Similar polyhedra are to one another in the triplicate ratio of their homologous edges.

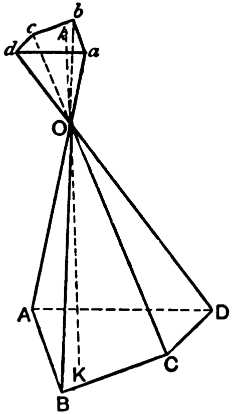
Let the smaller of two polyhedra be placed within the other so that its edges and faces are parallel to the corresponding edges and faces of the other, then the lines joining corresponding corners will meet in a point within the polyhedra and the planes through this point and corresponding edges will divide the polyhedra into the same number of similar pyramids.

Let O be the common vertex and $ABCD$, $abcd$ the bases of two of these pyramids, a, b, c, d being on OA, OB, OC, OD respectively;

then $ABCD$, $abcd$ being similar plane figures their areas are in the duplicate ratio of their homologous sides AB, ab:

also if the normal from O meet the plane $ABCD$ in K and the plane *abcd* in *k*, *OK*, *Ok* will be the heights of the two pyramids, and will be in the ratio of *OA* : *Oa* and therefore of *AB* : *ab*.





But the volumes of the two pyramids are in the ratio compounded of the ratios of their bases and their heights, and therefore in the ratio compounded of the duplicate ratio of *AB* : *ab* and of the simple ratio of *AB* : *ab*, that is, in the triplicate ratio of the homologous sides *AB*, *ab*.

The same being true for each pair of pyramids, of which the polyhedra are composed, and the ratios of each pair of homologous sides being the same, the ratio of the sums of the pyramids, that is, of the volumes of the polyhedra is the triplicate ratio of their homologous edges. Q. E. D.

Ex. Three meat pies are made with the same paste, in similar dishes of different sizes; if the meat is duly proportioned to the crust in the middle-sized dish, the crust will be in excess in the smaller, and the meat in the larger pie.

APPENDIX.

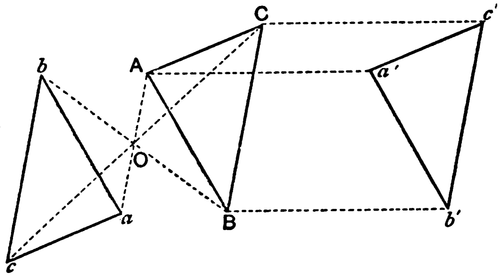
ON SYMMETRY IN A PLANE AND IN SPACE.

SYMMETRY consists in an exact correspondence of points, lines, or surfaces on opposite sides of a definite point or centre, or of a definite line or axis, or of a definite plane. We proceed to examine these different kinds of symmetry in some detail, first for figures in a plane, and then for figures in space.

SYMMETRY OF FIGURES IN ONE PLANE.

I. *Central or Point Symmetry.*

If A, a are two points on a straight line passing through a point O at equal distances from O on opposite sides of it, they are said to be symmetrical with respect to O. If A moves along any line, a moves along a corresponding line, and the two lines are symmetrical with respect to O, which is termed their centre of symmetry.



It is obvious that by turning Oa about O in the plane through two right angles, a will be brought into coincidence with A, and thus two lines or figures, which have central symmetry, are superposable by rotation about the centre *without leaving the plane.*

In figures, which have central symmetry, corresponding straight lines are parallel but have *opposite senses:* the same figures can be so placed that the corresponding lines are parallel and have *like senses*, and then the lines joining corresponding points are all parallel. This is obvious from the relations of the triangles ABC, abc, $a'b'c'$ in the figure.

As examples, the following simple cases may be noted:

Equal and parallel straight lines have central symmetry.

A parallelogram is a figure symmetrical with respect to the intersection of its diagonals.

A circle is symmetrical with respect to its centre.

II. *Axial or Line Symmetry.*

Two points A, a are said to be symmetrical with respect to a given line, when the line $A\,a$ joining them is at right angles to, and bisected by, the given line. If A moves along any line, a moves along a corresponding line, and the two lines are symmetrical with respect to the given line, which is termed their axis of symmetry. Two lines or figures, which have axial symmetry, can be brought into coincidence by turning the half of the plane in which one of them lies about the axis of symmetry till it coincides with the other half, but not (in general) by moving them about in their plane. In figures, which have axial symmetry, corresponding straight lines (if produced) meet on the axis of symmetry and are equally inclined to it, and lines joining corresponding points are all parallel. If without leaving

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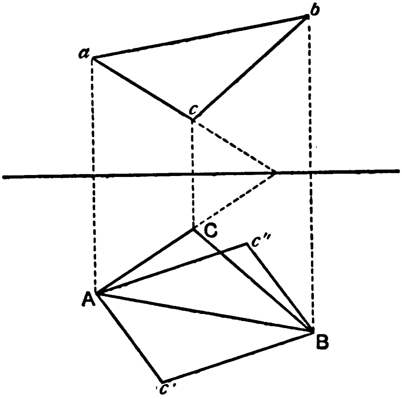
the plane two corresponding lines are brought into coincidence, the figures are not coincident, but symmetrical, with respect to the coincident lines or with respect to the perpendicular through their middle point, according as they are superposed with the same or with opposite senses.

A plane figure and its image by a plane mirror perpendicular to its plane are symmetrical with respect to the intersection of the mirror and the plane as the axis of symmetry; hence of two figures having axial symmetry one may be called the *geometrical image* of the other.



All the above statements will easily be seen to be true from the figure, in which *abc* is the geometrical image of *ABC* and *ABc'*, *ABc''* are the positions of *abc*, according as, without leaving the plane, *ab* or *ba* is made to coincide with *AB*.

The following are cases of axial symmetry, with or without central symmetry :—

1. An isosceles triangle with respect to the bisector of its vertical angle.

2. Rhombus—Symmetry with respect to either diagonal, also a centre of symmetry.

3. Rectangle—Two axes of symmetry, also a centre.

4. Square—Four axes of symmetry, also a centre.

5. Equilateral Triangle—Three axes of symmetry, no centre.

6. Regular Polygon of *n* sides—*n* axes of symmetry—also central symmetry, if *n* is even, but not if *n* is odd.

7. Circle—symmetrical with respect to every diameter, and also with respect to the centre.

8. A figure which is symmetrical with respect to every line through a given point is a circle.

9. A figure symmetrical with respect to two axes at right angles to one another is symmetrical with respect to their point of intersection.

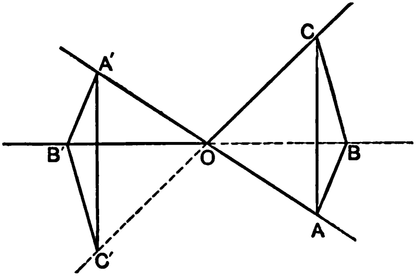
SYMMETRY IN SPACE.

There are three kinds of symmetry in space, namely, symmetry with respect to a *point*, to a *line*, and to a *plane*. The relations between these among themselves, and their analogy or contrast with the two kinds of symmetry in a plane may be advantageously studied in some detail.

I. *Central or Point Symmetry.*

This has already been considered, as regards solid angles, at the beginning of Section V., where it has been shown that opposite solid angles, which have symmetry with respect to their common vertex (unlike opposite plane angles) are not in general superposable.

Now consider a triangular pyramid *OABC*, and the pyramid *OA'B'C'* formed by producing the edges of the first through *O*, so that *OA'*, *OB'*, *OC'* are respectively equal to *OA*, *OB*, *OC*.



It is plain that these opposite pyramids have the faces of the one, each identically equal to a corresponding face of the other, the sides and planes of their bases *ABC*, *A'B'C'* parallel, and their dihedral angles equal; yet if, by turning about the normal through *O* to the plane *ABA'B'*, the face *A'OB'* is brought to coincide with *AOB*, the corners *C'*, *C* being on opposite sides of the plane, the pyramids will not coincide; while if by turning about the bisector of the angle *AOB'*, *OB'* is brought to lie upon *OA*, and *OA'* on *OB*, and *C'*, *C* to be on the same side of the plane, the faces *A'OB'*, *AOB* will not coincide, except in the case where $OA = OB$, so that, except in very special cases, the pyramids are not superposable.

The same will obviously be true of any two solid polyhedra, which have symmetry with respect to *O*, since they may be made up of the sums and differences of pyramids with their vertices at O.

II. *Axial or Line Symmetry.*

If the plane sections of a solid figure or figures by planes normal to a given axis have central symmetry, each with respect to the point where its plane meets the axis, the figure or figures are symmetrical with respect to the axis. It is obvious, just as in the case of central symmetry in a plane, that solid figures which have axial symmetry are superposable.

As instances may be mentioned:—

1. A cuboid with respect to an axis parallel to any of its edges through the intersection of its diagonals.

2. A right prism or pyramid on a regular polygonal base of an even number of sides, with respect to the normal to the base through its centre.

3. A right cone or cylinder with respect to its axis.

4. Any surface of revolution with respect to its axis.

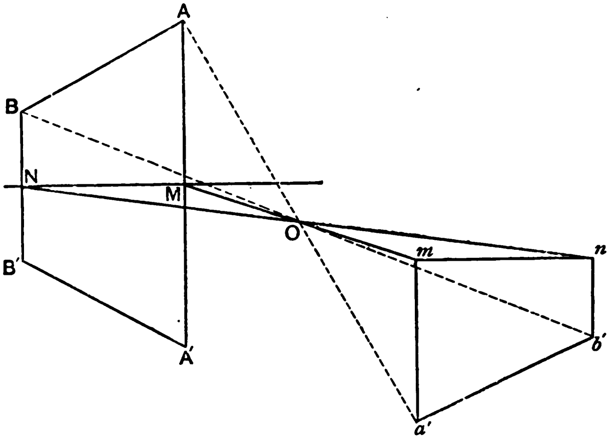
5. A sphere with respect to any diameter.

6. A regular tetrahedron with respect to the line joining the middle point of two opposite edges.

III. *Symmetry with respect to a Plane.*

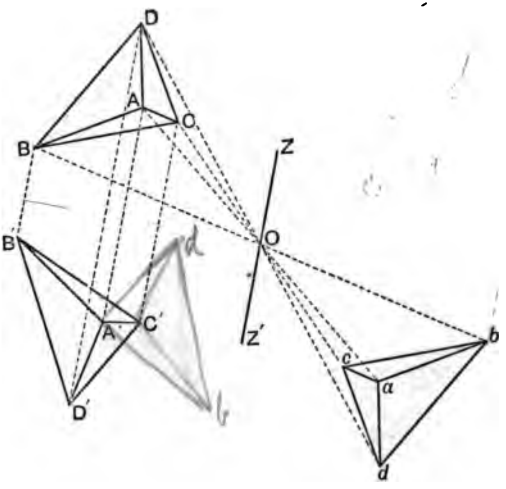
Two points on a normal to a plane at equal distances from it on opposite sides are said to be symmetrical with respect to the plane, and if every point in each of two figures is thus symmetrical with respect to a corresponding point in the other, the figures are said to be symmetrical with respect to the plane. This is the familiar case of an object and its image in a plane mirror, so that of two such figures each may be said to be the geometrical image of the other. It is obvious that a solid figure and its geometrical image are not *superposable*, the one on the other, although all corresponding lines and angles are equal. Also that corresponding planes and lines are equally inclined to the plane of symmetry, meeting in that plane in the same line or point. Such figures are however *opposable*. In other words—

Solid figures, which are symmetrical with respect *to a plane* (or each of which is the *geometrical image* of the other) can be so placed as to be symmetrical with respect to a *point*, and *vice versâ*.



For let A, A' be two corresponding points in the two solids, and let M be the common projection of A, A' on the plane of symmetry and therefore the middle point of A, A', and let the solid, of which A' is one point, be turned through half a revolution about the normal to the plane of symmetry at O, any point in that plane; then, by the rotation supposed, M would be brought to m in MO produced, so that $Om = OM$, and A' to a' in the normal to the plane through m, so that ma', MA' are equal and in the same sense, and Aa' will pass through, and be bisected, in O. Hence all corresponding points in the first solid and the second in its new position will pass through, and be bisected in O, that is, the two figures will now have central symmetry with respect to O.

In the annexed figure $A'B'C'D'$ represents a tetrahedron which is symmetrical with $ABCD$ with respect to a *plane*



through O normal to AA', and $abcd$ is one symmetrical with $ABCD$ with respect to the *point* O: neither $A'B'C'D'$ nor $abcd$ can be superposed on the tetrahedron $ABCD$, but either can be superposed on the other by half a revolution about the normal ZOZ' to the plane of symmetry of $ABCD$, $A'B'C'D'$. In other words—

$ABCD$, $A'B'C'D'$ have symmetry with respect to a plane.
$A'B'C'D'$, $abcd$ „ „ „ an axis (ZOZ')
$abcd$, $ABCD$ „ „ „ a point O.

A familiar example of the symmetry of non-superposable figures is afforded by a pair of outstretched hands. They can be so placed, palm opposite to palm, that each is the *image* of the other: and then by turning one of them round through half a revolution, keeping the palms opposite, the lines joining the tips of corresponding figures meet and bisect one another in the same point, so that they then have central symmetry. They are not, however, superposable; the right-hand glove will not fit the left-hand, unless it is turned inside out.

From the foregoing discussion it appears that the distinction between superposable and opposable figures, which for solid figures is a fundamental one, does not hold good for figures in one plane, since, if they are opposable, they are also superposable without leaving the plane; but that of figure and geometrical image holds, since these are not superposable except by moving one out of the plane and turning it over. This suggests that, if we could conceive a solid figure moved out of three-dimensional into four-dimensional space, some analogous process of turning over or inside out might render opposable solid figures superposable. Be this as it may, it seems desirable in three-dimensional geometry explicitly to *assume*, as we have done in our postulate, the equality of volume of opposable solids as well as of superposable. Otherwise the proof of the equality of volume of opposable solids or solid angles depends on our devising some method of dividing them into parts and rearranging the parts so as to make them superposable, as in the alternative proof of Th. XXIX., and this can only be done, except in special cases, by subdivision into infinitely small parts. It is true that this infinite subdivision is necessary for the proof of Th. XXXI., but in cases which do not depend on this theorem the directness of proof obtainable by the assumption of the postulate as opposed to the artificial treatment otherwise necessary seems a sufficient reason for admitting it among the recognised Postulates of Geometry.

A fuller development of the subject of Symmetry is to be found in treatises on Crystallography.

As additional instances of symmetry of the various kinds discussed above examine the following:—

1. A parallelepiped has a centre of symmetry, and any plane through its centre of symmetry divides it into equal parts, which may be so placed as each to be the image of the other.

2. A sphere has central symmetry, symmetry with respect to every line through the centre, and also with respect to every plane through the centre.

3. A regular tetrahedron has planes and axes of

symmetry, but no centre of symmetry. How many planes? and how many axes?

4. A cube and a regular octahedron have both centre, axes, and planes of symmetry. How many of each?

SECTION VI.

SPHERICAL SURFACE GEOMETRY.

THE present section will be devoted to the discussion of the simpler and more elementary properties and relations of lines and figures on a spherical surface. The study of the angular relations of lines and planes in space is greatly facilitated by measuring the angles, plane, dihedral, or solid, by means of arcs, angles, or areas on the surface of a sphere. The full development of these relations is the object of Spherical Trigonometry: here only a few of the more general relations which lie at the basis of Spherical Trigonometry will be established, not so much as preliminary to the study of that subject as with a view to a comparison with the corresponding propositions in Plane Geometry—a comparison which will be found, both by the resemblances and contrasts which it presents, to throw instructive light on the essential nature and peculiarities of Plane as distinguished from Surface Geometry generally.

For this purpose it will not be necessary to give the various propositions and proofs with all the formality of the previous sections, and so, for the sake of conciseness, only such outlines or hints for the demonstrations will be given as appear to be needed to enable the student to develop them more fully for himself.

the surface require more than two po
determination.

Farther assuming[1] that between t
face there is (in general) *only one* lin
that can be drawn between them, t
great circle through those points ; s
there would be another on the other
symmetrical with it and therefore equ

Thus the great circle on a spherica
analogous to the straight line on a pla
determined by two points, and so bei
between those points.

Hence, calling that line on any
pletely determined by two points th
surface, the great circle is the *dir*
surface, as the straight line[2] is the
surface. The lesser arc of the gre
given points in it, being the least dist
points, may be termed the *direct*
term *direct line* being reserved for th

In the exceptional case, where th
extremities of a diameter, any numb
be drawn through them, and there is
of lines less than any other that c
them, the length of any one of these
circumference of a great circle.

There is, of course, no such exce
surface.

The extremities of the diameter n

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[1] Cf. Preliminary Discussion, § 7.

[2] It should be remembered also that th
line for three-dimensional space.

I.—*The Great Circle the " Direct Line " of the Spherical Surface.*

Adopting the definition of p. 73:

Def. A spherical surface is the surface locus of a point whose distance from a fixed point is equal to a given distance. The fixed point is its *centre*, any line from the centre to the surface a *radius*, and every line through the centre terminated both ways by the surface a *diameter*.

It follows that any portion of a spherical surface may be moved freely to another position on the surface without leaving the surface and without any change of form or size.

Also that the surface is *symmetrical* both with respect to the centre, with respect to any diameter, and with respect to any plane through the centre.

The intersection of a spherical surface and a plane is the locus of a point in the plane, whose distance from the centre of the sphere is equal to the radius, and this locus is a circle whose centre is the foot of the normal to the plane from the centre and whose radius is less as the length of that normal is greater.

Hence the greatest plane sections of a spherical surface are those with planes passing through the centre. They are termed *great circles* of the sphere, and they have radii equal to that of the sphere. Sections by planes which do not pass through the centre are termed *small circles*, and the radius of a small circle may be of any length less than that of the sphere.

A great circle is completely defined by two points on the surface, provided that they are not the extremities of the same diameter; while small circles and all other lines on the surface require more than two points for their complete determination.

Farther assuming[1] that between two points of the surface there is (in general) *only one* line less than any other that can be drawn between them, that line must be the great circle through those points; since for any other line there would be another on the other side of the great circle symmetrical with it and therefore equal to it.

Thus the great circle on a spherical surface is completely analogous to the straight line on a plane in being completely determined by two points, and so being the shortest distance between those points.

Hence, calling that line on any surface which is completely determined by two points the *direct line* for that surface, the great circle is the *direct line* of a spherical surface, as the straight line[2] is the *direct line* of a plane surface. The lesser arc of the great circle between two given points in it, being the least distance between the two points, may be termed the *direct arc* between them, the term *direct line* being reserved for the whole circle.

In the exceptional case, where the two points are the extremities of a diameter, any number of great circles may be drawn through them, and there is thus an infinite number of lines less than any other that can be drawn between them, the length of any one of these lines being the semi-circumference of a great circle.

There is, of course, no such exceptional case on a plane surface.

The extremities of the diameter normal to the plane of any great circle are termed its *poles*, and the direct arc from either pole to any point in the great circle is a quadrant of a great circle.

---

[1] Cf. Preliminary Discussion, § 7.

[2] It should be remembered also that the straight line is the *direct line* for three-dimensional space.

II.—*Two Great Circles—Lunes.*

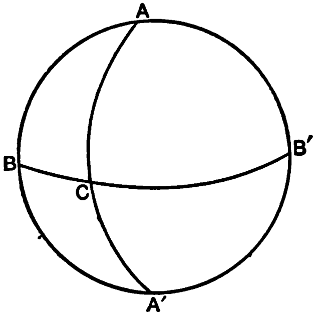
Since the planes of any two great circles intersect in a diameter, the circles intersect at two points at the extremities of that diameter. Hence any two direct lines on a sphere meet in two points and bisect one another at those points. Thus they are unlike straight lines in a plane which either meet in one point only or are parallel. There are no parallel direct lines on a spherical surface.

The angle between the two circles where they meet is the angle between the tangents to the circles at that point, and, these being perpendicular to the diameter, which is the intersection of the planes of the circles, the angle between the circles at either point where they meet is the dihedral angle between their planes.

Two great circles divide the whole spherical surface into four parts, called *lunes*. It is obvious that two lunes which have equal angles can be brought into coincidence and so are equal, whence it follows that the area of a lune is proportional to its angle, so that it is the same part of the whole surface that its angle is of four right angles.

The great circle of which the angular points of a lune are the poles bisects the lune and its bounding arcs, and the part of it intercepted within the lune measures the dihedral angle between the planes of the bounding arcs, and so also measures the angle of the lune.

Measuring the angle A of the lune between ABA', ACA' by its ratio to a right angle, and the arc BC intercepted on the circle of which A, A' are the poles by its ratio to a quadrant, the measures of the angle and the arc are equal ratios, and we may say, for the sake of brevity, that the angle A is equal to the arc BC.



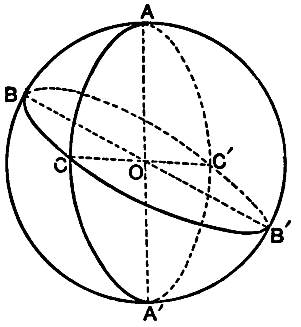
It is obvious that all the properties of angles at a point in a plane (Pl. G. i. 2–4, or Euc. i. 13–15) are true for angles at a point on a spherical surface, direct lines or great circle taking the place of straight lines.

III.—*Three Great Circles. Spherical Triangles.*

If three great circles are drawn on a spherical surface, they divide the sphere into eight regions, each of which is bounded by three direct arcs, and is called a *spherical triangle.*

If ABC is one of these triangles, and A', B', C' the opposite ends of the diameters through A, B, C respectively, the triangle $A'B'C'$ has its sides $B'C'$, $C'A'$, $A'B'$ respectively equal to the sides BC, CA, AB, and its angles A', B', C' to the angles A, B, C; and farther for every point P

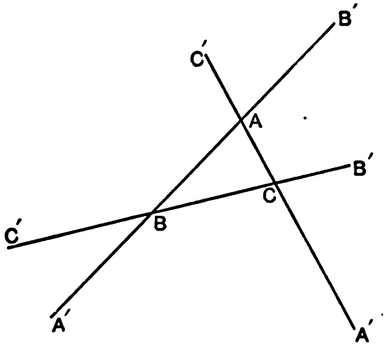
in the triangle ABC there is a corresponding point P', the opposite end of the diameter through P, exactly similarly related to the triangle $A'B'C'$. Nevertheless, as we have seen that the solid angle $O(A'B'C')$ is not superposable on the solid angle $O(ABC)$, in the same way the spherical triangle $A'B'C'$ is not superposable on ABC. The triangle $A'B'C'$ may be termed the *opposite* to ABC, and any triangle that can be made to coincide with $A'B'C'$ may be said to be *opposable* to ABC. Then, in accordance with the postulate assumed in Section V., we shall assume as a postulate of Spherical Surface Geometry the following:



POST. *Opposite or opposable spherical triangles are equal.*

The eight associated triangles into which three great circles divide a sphere may then be regarded as four pairs of equal and opposite triangles: and the relation between either triangle of one pair and either of another pair is that, while one side and the angle opposite to it are the same in both, the other two sides and the angles opposite to them in the one are supplementary to those in the other.

A plane is divided by three straight lines, meeting in the points *A*, *B*, *C*, into *seven* regions : one the finite triangle *ABC*, three infinite regions bounded by two of the lines (as *B'AC'*), three bounded by three, as *A'BCA'*. If a plane be regarded as the limit of a spherical surface, when its centre is at an infinite distance and therefore its radius infinite, we may say that



the triangle *A'B'C'* opposite to *ABC* is wholly at infinity, that the triangles *AB'C'*, *BC'A'*, *CA'B'* have one side wholly at infinity, and that the triangles *BA'C*, *CB'A*, *AC'B* have one vertex wholly at infinity, the opposite infinite triangles as *AB'C'*, *A'BC* being equal, and the finite triangle *ABC* zero compared with the infinite spherical surface.

It will be observed that with the definition given above of a spherical triangle, each side is less than half a direct line or the semi-circumference of a great circle, and each angle less than two right angles. If this restriction were removed, as, for instance, if one side of the triangle *ABC* were taken to be *AB'A'B* instead of the direct arc *AB*, so that the triangle would have a re-entrant angle at *C*, the propositions we shall hereafter prove would not all be necessarily true. Thus in the supposed case the two sides AC, CB would be less, not greater, than the third.

The sides of the triangle ABC measure the angles in the plane faces of the trihedral angle $O(ABC)$, and its angles measure the dihedral angles between the faces, and the triangle itself measures the solid angle.

Each side of a spherical triangle (being the direct line between two points, is less than any other lines connecting those points, and so) *is less than the sum of the other two sides.*

This is equivalent to, and is a proof of, Th. XVIII.

The sum of the three sides is less than the circumference of a great circle.

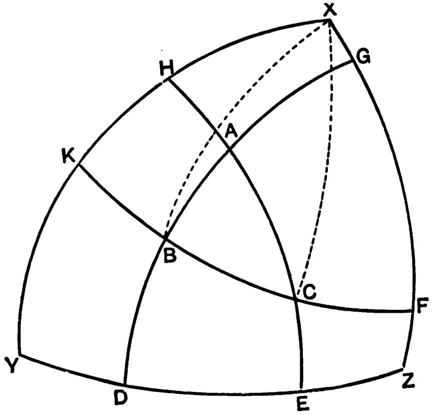
For BC being less than $A'B + A'C$,
$BC + CA + AB$ is less than $A'BA + A'CA$ or two semicircumferences.

This is equivalent to, and is a proof of, Th. XIX. for the case of a trihedral angle.

Ex. Extend the proof to the case of a spherical polygon, subtending a polyhedral angle.

IV.—*The Polar Triangle.*

On AB, AC, two sides of the triangle ABC, take the quadrants AD, AE, and similarly on BC, BA, take the quadrants BF, BG, and on CA, CB the quadrants CH, CK: then A, B, C are respectively the poles of the direct lines through D and E, F and G, H and K. Then since B is the pole of the line FG, and C that of the line HK, if FG and KH intersect in X, BX and CX are both quadrants, and therefore X is the pole of BC. Similarly if HK and ED intersect in Y, Y is the pole of CA, and if DE and GF intersect in Z, Z is the pole of AB. Thus the triangles ABC, XYZ are so related that, A, B, C being the poles of YZ, ZX, XY, respectively, X, Y, Z are the poles of BC, CA, AB, and each is then said to be the *polar triangle* of the other.



Since X is the pole of BC,
the angle $X = KF = KC + BF - BC$,
so that, KC, BF being quadrants, the arc measure of X and the arc BC are supplementary.

Also, since A is the pole of YZ,
the angle $A = DE = YE + ZD - YZ$,
so that YE, ZD being quadrants, the arc measure of A and the arc YZ are supplementary.

Similarly the angles Y, Z are supplementary to CA, AB, and the angles B, C supplementary to ZX, XY respectively.

Hence *the sides of the polar triangle are the supplements of the angles of the original triangle, and the angles of the former the supplements of the sides of the latter.*

It should be observed that we have taken X on the same side of BC as A, Y on the same side of CA as B, and Z on the same side of AB as C. If X', Y', Z' are the poles opposite to X, Y, Z, of the triangles associated with XYZ all but $X'Y'Z'$ have two sides, and the angles opposite to them equal to two angles and the sides opposite to them in the triangle ABC, and the third side and third angle only supplementary to the third angle and third side. Thus $X'YZ$ has the angles at Y, Z equal to AC, AB respectively, and the sides $X'Y$, $X'Z$ equal to the angles C, B respectively, but the angle X' supplementary to BC, and the side YZ supplementary to A. It may in fact be regarded as the polar triangle of $A'BC$.

V.—*Spherical Excess.*

We can now prove that the *sum of the angles of a spherical triangle is greater than two right angles.*

For the sum of the angles of the triangle and the sides of the polar triangle is equal to six right angles, and the sum of the sides of the polar triangle being less than four right angles, the sum of the angles of the triangle is greater than two right angles.

This is equivalent to, and is a proof of, Th. XX.

The sum of the angles of a spherical triangle may have any value between two and six right angles.

Since the sum of the angles of a plane triangle is two right angles, the excess of the sum of the angles of a spherical triangle over two right angles is termed its *spherical excess.*

The spherical excess may have any value less than four right angles.

Measuring angles by their ratios to a right angle, and E denoting the spherical excess of the triangle ABC,

$$E = A + B + C - 2.$$

Ex. 1. Show that E^A being the spherical excess of $A'BC$ or $AB'C'$, $E_A = 2A - E$.

2. Show that the spherical excess of a triangle is less than double its least angle.

VI.—*Area of a Spherical Triangle.*

Since the triangle ABC is equal to $A'B'C'$ (Fig. p. 114), the sum of the triangles CAB, $CA'B'$ is equal to the lune whose angle is C. Also the sum of the triangles ABC, $A'BC$ is equal to the lune whose angle is A, and the sum of the triangles ABC, $AB'C$ is equal to the lune whose angle is B.

Hence, observing that the four triangles ABC, $A'BC$, $A'CB'$, $AB'C$ make up the surface of a hemisphere, the sum of twice the triangle ABC and the hemispherical surface is equal to the sum of the lunes whose angles are A, B, C.

Therefore, observing that the area of the lune whose angle is a right angle is $\frac{1}{4}$ of the surface of the sphere,

$$\frac{2 \times \text{the area of } ABC}{\frac{1}{4} \times \text{surface of the sphere}} + 2 = A + B + C,$$

the angles A, B, C being measured by their ratio to a right angle;

so that

$$\frac{\text{the area of a spherical triangle}}{\text{surface of the sphere}}$$

$$=\frac{\text{the spherical excess of the triangle}}{\text{eight right angles}}.$$

In a quadrantal triangle—that is, one whose sides are quadrants—the angles are right angles and the spherical excess is a right angle; whence its area = $\frac{1}{8}$ × the surface of the sphere, and

$$\frac{\text{the area of a spherical triangle}}{\text{the area of a quadrantal triangle}}$$

$$=\frac{\text{the spherical excess of the triangle}}{\text{a right angle}};$$

so that, if the unit of area be that of a quadrantal triangle, and the angular unit a right angle, we may say that

(*the measure of*) *the area of a spherical triangle* = (*that of*) *its spherical excess.*

The above is the equivalent of Th. XXII., a trihedral angle being measured by the triangle which it subtends on the surface of a sphere whose centre is the vertex of the trihedral angle.

It should be observed that the areas here spoken of are not *absolute* areas,[1] but areas relatively to the given spherical surface. Their absolute value depends on the magnitude of the sphere.

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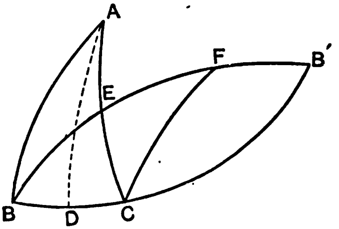
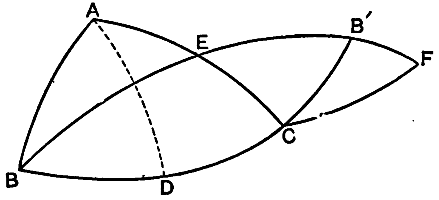
[1] Observe also that on a given spherical surface the circumference of a great circle affords a *natural* unit of length, and the area of the sur-

VII.—*Properties of a Spherical Triangle compared with those of a Plane Triangle.*

1. In a plane triangle an exterior angle is greater than either of the interior opposite angles. This is not always the case in a spherical triangle.





Let the side *BC* of the triangle *ABC* meet the direct line through *B* and *E*, the middle point of *AC*, in *B′*, and on

---

face a *natural* unit of area, just as a right angle is a *natural* unit of angle. On a plane surface, of course, there is no *natural* unit of length or area.

EB' take EF equal to EB: then according as BE is less or greater than a quadrant, F will fall within the angle ACB' or beyond it. Now if the triangle AEB is turned about E through two right angles, A will come to C and B to F, and the angle EAB will coincide with ECF: hence as BE is less than, equal to, or greater than, a quadrant, the exterior angle ACB', is greater than, equal to, or less than, ECF, that is, than the interior angle A opposite to the side BC which is produced.

The same is true for the exterior angle BCA' (and therefore for ACB' which is equal to it), and the interior angle B, according as AD is less than, equal to, or greater than, a quadrant, D being the middle point of B. Hence,

The exterior angle of a spherical triangle is greater than, equal to, or less than, either of the interior opposite angles, according as the direct arc from the other to the middle point of the opposite side is less than, equal to, or greater than, a quadrant.

Observe that in the case of a plane triangle F could not fall beyond BCB', because two straight lines which meet at one point (B) cannot meet again.

Ex. 1. If the alternate angles, which AC makes with BAB', BCB', are equal, the semi-circumference BEB' through the middle point E of AC is bisected in E. Deduce from this a proposition in plane geometry.

Ex. 2. The exterior angle $ACB' = A$, if BC and BA are supplementary, and conversely.

Ex. 3. If two angles of a triangle are supplementary, the sides opposite to them are supplementary.

Ex. 4. If the angles adjacent to a given side of a triangle are supplementary, the locus of the vertex is a great circle. What fact in plane geometry is deducible?

Ex. 5. If the angle opposite to a given side is supplementary to one of those adjacent to that side, the locus of the vertex is a small circle.

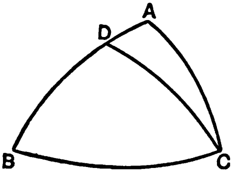
2. *If two sides of a triangle are equal, the angles opposite to them are equal: and conversely.* (Cf. Th. XXI.)

In the triangle ABC let AB be equal to AC; then each is equal to both $A'B'$ and $A'C'$ of its opposite triangle $A'B'C'$, and the triangle ABC may be so moved on the spherical surface that AC comes to coincide with $A'B'$ and AB with $A'C'$, and therefore BC with $C'B'$. Hence the angle B coincides with and is equal to the angle C', but the angle C' is equal to C, therefore the angle B is equal to the angle C.

For the converse, if the angles B, C are equal, in the polar triangle the sides ZX, XY are equal, and therefore, by what has just been proved, the angles Y, Z are equal, whence the sides AC, AB are equal.

The simplest proof of this proposition for a plane triangle consists in showing that, if two isosceles triangles coincide, they can be made to coincide after turning one of them over and replacing it on the other. (Cf. Pl. G. I. 6.) This, involving the moving of the triangle out of its plane, is inapplicable to the case of the spherical triangle, since when thus turned over, if the angular points were brought into coincidence, the arcs between them would not coincide. On the other hand the proof here given for the spherical triangle is inapplicable to the plane triangle, since it has no opposite triangle, at least at a finite distance from it.

3. *If two angles of a triangle are unequal, the sides opposite to them are unequal, that being the greater which is opposite to the greater angle: and conversely.*



In the triangle ABC let the angle ACB be greater than ABC, and let the angle BCD be equal to ABC.

Then $BD = CD$, and therefore $BA = CD + DA$, and therefore $> CA$.

Hence, according as the angle B is greater than, equal to, or less than, C,

so is AC greater than, equal to, or less than, AB:

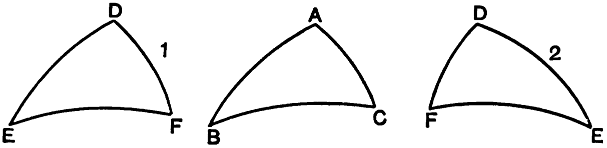
and therefore conversely, as AC is greater than, equal to, or less than AB,

so is the angle B greater than, equal to, or less than, C.

The proof of this proposition usually given in plane geometry (Pl. G. I. 11, 12, or Euc. I. 18, 19) is not applicable to spherical geometry, since it depends on the fact of the exterior angle of a triangle being greater than the opposite interior angle.

The proof here given however is applicable to the plane triangle, if we admit the postulate that there is *not more than one* line joining two points in a plane which is shorter than any other line or lines joining them.

VIII.—*Different conditions under which triangles are equal in all respects.*



Spherical triangles are equal in all respects either if they are superposable, as *ABC*, *DEF* (Fig. 1), or if they are opposable, that is, the one superposable on the triangle opposite to the other, as *ABC*, *DEF* (Fig. 2).

Different conditions under which this is true are the following:—

1. *If two sides and their included angle in the one are equal to corresponding parts in the other.* (Pl. G. I. 5, or Euc. I. 4.)

The proof is identical with that for plane triangles by superposing *DEF* (Fig. 1) on *ABC*, and *DEF* (Fig. 2) on *A'B'C'*, the opposite to *ABC*.

The remarks in small type on VII. 2 apply here also.

2. *If two angles and their common arm in the one are equal to corresponding parts in the other.* (Pl. G. I. 6, or Euc. I. 26, first case.)

The proof is by superposition on *ABC* or *A'B'C'*, as in the last case. (Cf. Pl. G. I. 6.)

Ex. Prove this also by means of the polar triangle.

3. *If the sides of the one are severally equal to those of the other.* (Pl. G. I. 18, or Euc. I. 8.)

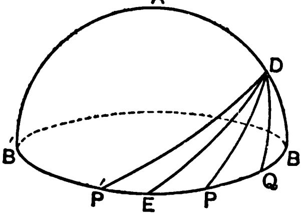
Make one side, as *DF*, coincide with the corresponding side, *AC* or *A′C′*, of *ABC* or *A′B′C′*, so that the vertices *E* and *B* or *E* and *B′* lie on opposite sides of *AC* or *A′C′*. Then the proof is the same as that with the corresponding construction for plane triangles. (Cf. Pl. G. I. 18, second proof.)

4. *If the angles of the one are severally equal to the angles of the other.*

In this case the sides of the polar triangle of the one are equal severally to the sides of that of the other, therefore by the last case their angles are equal, and therefore the sides of the original triangles are equal.

There is of course no corresponding proposition in plane geometry, the three angles of a plane triangle not being independent, as they are in a spherical triangle, but connected by the condition that their sum is two right angles. In a plane triangle, shape (determined by the angles) and size (determined by the sides) are independent, but in a spherical triangle the shape determines the size, and the size determines the shape.

IX.—*Greatest and least distances of a point from a given great circle.*



Let A be the pole of a given great circle and D a given point, and let the direct line (or great circle) through A and D cut the given great circle in B, B', and let E be the pole of BDB', so that E is on the given great circle and bisects the semi-circumference between B and B'. Then if P be a point on the great circle between D and E, the exterior angle DPE is greater than the interior angle DBP, since the direct arc from D to the middle point of BP is less than a quadrant (the circle of which D is the pole lying, on this side of E, below BEB'): therefore DBE being a right angle, DPE is an obtuse angle and DPB an acute angle, so that DP is greater than DB and less than DB'.

Also if P' be a point in EB', the direct arc from D to the middle point of $B'P'$ being greater than a quadrant, the exterior angle $DP'B$ is less than $DB'E$, and is therefore acute, so that $DP'B'$ is obtuse, whence, as before, DP' is greater than DB and less than DB'. Hence DB, DB' are respectively the least and greatest distances from D to the given circle.

Also if Q be a point between B and P, since DQP is obtuse and DPQ acute, DP is greater than DQ.

Further, from the symmetry with respect to the plane BAB', for every line DP there is an equal line Dp on the other side of BAB', such that $Bp = BP$, and there can be no other line equal to DP.

Hence we have proved the following proposition :—

Of all the direct arcs that can be drawn on a spherical surface to a given direct line on it from a given point not in that line, the two direct arcs perpendicular to the given line are, the greater of them the greatest, and the less the least:

and of the others those which are drawn to points at equal distances from the foot of either perpendicular are equal, and that drawn to a point nearer to the foot of the least arc is less than that drawn to a point farther from it.

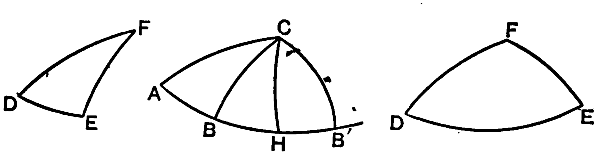
Compare this with Theorem XIV., and also with the corresponding proposition in plane geometry (Pl. G. I. 15).

Ex. 1. Prove that if $EP = EP'$, the angles DPB and $DP'B$ are equal, and the sides DP, DP' supplementary.

Ex. 2. Hence prove VIII. 4 directly without the aid of the polar triangles.

X.—*Ambiguous Cases.*

1. *If two triangles have two sides of the one respectively equal to two sides of the other and the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal.* (Cf. Pl. G. I. 20. In Euclid, this case, wanting in Book I., is implicitly treated in VI. 7.)



In the triangles ACB, DFE let AC, CB be respectively equal to DF, FE, and the angles A, D equal, then by superposing DFE on ACB so that DF coincides with AC and DE falls along AB, E must coincide either with B or with B', a point on the other side of the perpendicular CH such that HB' is equal to HB, since (by IX.) no other line from C can be equal to CB or FE.

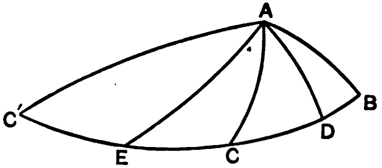
In the former case the triangles coincide : in the latter $AB'C$ is equal to CBB', and therefore supplementary to ABC, so that DEF is supplementary to ABC.

For the discussion of the additional condition, which in a particular case decides between the two possibilities thus proved, the student is referred to treatises on spherical trigonometry, especially that by McClelland and Preston, Part I., Arts. 70-75.

2. *If two triangles have two angles of the one respectively equal to two angles of the other and the sides opposite to one pair of equal angles equal, then the sides opposite the other pair of equal angles are either equal or supplementary, and in the former case the triangles are equal.*

This follows from the preceding proposition applied to the polar triangles.

The annexed figure shows how the ambiguity arises.



AD is the perpendicular on BC from A, DE a quadrant, and therefore E the pole of AD. If $EC = EC$, $ACB = AC'B$ and AC' is supplementary to AC (cf. IX. Ex. 1), so that the triangles ABC, ABC' have the side AB and the angle B common, and the angle $ACB = AC'B$; but they are unequal.

For the corresponding proposition in plane geometry, see Pl. G. I. 19, or Euc. I. 26, second case. The ambiguity ceases, when the spherical surface becomes plane by the radius of the sphere becoming infinite, since ABC remaining finite, E and therefore C' goes off to an infinite distance.

XI. Since on a spherical surface all direct lines (or great circles) intersect without exception, there are no propositions directly corresponding to the propositions involving parallels in plane geometry, and the classification of quadrilaterals as parallelograms, rectangles, and squares is inapplicable, so that the propositions as to areas in Euclid from I. 35 to the end of Book II., or Pl. G. Book II., have no direct analogues in Spherical Geometry.

Farther, since shape and size are not independent on a spherical surface, the doctrine of similar figures has no place in Spherical Geometry.

The analogue to a circle in Plane Geometry is a small circle of the sphere in Spherical Geometry. It will be a useful exercise for the student to examine, as to the different properties of the circle proved in elementary plane geometry, which have, and which have not, directly corresponding propositions in Spherical Surface Geometry.

The farther geometry of a spherical surface may be best studied in connection with Spherical Trigonometry. For this the student may be referred to Pt. II. of McClelland and Preston's *Treatise on Spherical Trigonometry.*

THE END.

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